ERASMUS IDS Course

[illegible], January 7

Subject: The concept of Conditional Distribution and Conditional Expected Value – part 2.

Introduction

In stochastic methods, for instance in Statistics, Econometrics the general problem has a following form:

for given R.V. $(X_1, X_2, .., X_n)$ we are looking for the interaction between the coordinates $X_1, .., X_n$ and we try to establish the measure of this interaction.

To do this we can use: the covariance value, the correlation coefficient, the notion of stoch. independence and others.

Among them is also the concept of conditional distribution and conditional expected value.

Of course we are going to explain these concepts only to the case $n=2$, so (X, Y) with finite discrete joint distribution.

The concept of conditional distribution & cond. exp. value

Suppose we have a R.V. (X,Y) and its joint distr $\mathbb{P}$, so

$$(X,Y) \longrightarrow \mathbb{P} = [p_{ij}]_{n \times m},$$

where $n = |X(\Omega)|$, $m = |Y(\Omega)|$.

Then the marginal distributions are as follows:

$$d_X : p_{i\cdot} = \sum_{j=1}^{m} p_{ij}, \quad i = 1, \ldots, n$$

$$d_Y : p_{\cdot j} = \sum_{i=1}^{n} p_{ij}, \quad j = 1, 2, \ldots, m,$$

where

$$p_{i\cdot} = P(\{\omega \in \Omega : X(\omega) = x_i\})$$

$$p_{\cdot j} = P(\{\omega \in \Omega : Y(\omega) = y_j\}).$$

Def 1.

By conditional distribution of X with respect of condition that $Y = y_k$ or of Y with respect of condition that $X = x_l$ we understand respectively

$$\frac{p_{ik}}{p_{\cdot k}} = \frac{P(\{\omega\in\Omega : X(\omega)=x_i \,\&\, Y(\omega)=y_k\})}{P(\{\omega\in\Omega : Y(\omega)=y_k\})}, \text{ for } i=1 \dots n$$

and fixed k, $k=1,2 \dots m$

or

$$\frac{p_{ik}}{p_{i\cdot}} = \frac{P(\{\omega\in\Omega : X(\omega)=x_i \,\&\, Y(\omega)=y_k\})}{P(\{\omega\in\Omega : X(\omega)=x_i\})}.$$

So,
$$\frac{p_{ik}}{p_{\cdot k}} = P(\{\omega\in\Omega : X(\omega)=x_i\} \,|\, \{\omega\in\Omega : Y(\omega)=y_k\})$$

and
$$\frac{p_{ik}}{p_{i\cdot}} = P(\{\omega\in\Omega : Y(\omega)=y_k\} \,|\, \{\omega\in\Omega : X(\omega)=x_i\})$$

Remark. Since
$$\sum_{i=1}^{n} \frac{p_{ik}}{p_{\cdot k}} = \frac{1}{p_{\cdot k}} \sum_{i=1}^{n} p_{ik} = \frac{p_{\cdot k}}{p_{\cdot k}} = 1$$

and
$$\sum_{k=1}^{m} \frac{p_{ik}}{p_{i\cdot}} = \frac{1}{p_{i\cdot}} \sum_{k=1}^{m} p_{ik} = \frac{p_{i\cdot}}{p_{i\cdot}} = 1,$$

the sequences given in def 1 define the probabilities distributions.

Having such distributions we can calculate the corresponding mathematical expectations which we denote as follows:

$$(\#1)\quad E(Y \mid X = x_i) = \sum_{k=1}^{m} y_k P(\{\omega\in\Omega : Y(\omega) = y_k\} \mid \{\omega\in\Omega : X(\omega) = x_i\}) \quad (i = 1, \dots, n)$$

$$= \sum_{k=1}^{m} y_k \frac{p_{ik}}{p_{i\cdot}} \, ,$$ and similarly

$$(\#2)\quad E(X \mid Y = y_k) = \sum_{i=1}^{n} x_i P(\{\omega\in\Omega : X(\omega) = x_i\} \mid \{\omega\in\Omega : Y(\omega) = y_k\})$$

$$= \sum_{i=1}^{n} x_i \frac{p_{ik}}{p_{\cdot k}} \, , \quad (k = 1, \dots, m)$$

which are called the <u>conditional math. expectations</u>.

Now for R.V. (X, Y) as above let the set

$$O = \{ (x_i, y_k) \in \mathbb{R}^2 \, , \, i = 1, 2, \dots, n \, , \, k = 1, 2, \dots, m \}$$

be a result of observation of $\omega \to (X(\omega), Y(\omega))$.

Then the set

$$A = \{ (x, y) \in \mathbb{R}^2 : x = x_i \, , \, y = E(Y \mid X = x_i) \, , \, i = 1, \dots, n \} ,$$

and similarly

$$B = \{ (x, y) \in \mathbb{R}^2 : y = y_k \, , \, x = E(X \mid Y = y_k) \, , \, k = 1, \dots, m \}$$

is called the regression curve of the first type Y with respect X on X with respect of Y.
These curves are used for the determination of relationships between X and Y, what is important in econometrics and in general in statistics.

Now we show how it works on example.

Example.

We have

$$(X,Y) \longrightarrow P = \begin{array}{c|cc} & -1 & 1 \\ \hline 1 & \frac{11}{50} & \frac{1}{10} \\ 2 & \frac{8}{50} & \frac{13}{50} \\ 3 & \frac{3}{50} & \frac{1}{5} \end{array},$$

where $X(\Omega) = \{1,2,3\}$, $Y(\Omega) = \{-1,1\}$,

so $d_X = \left(\frac{16}{50}, \frac{21}{50}, \frac{13}{50}\right)$

$d_Y = \left(\frac{22}{50}, \frac{28}{50}\right)$.

We shall find the regression curve of Y with resp. X and vice-versa.

First we need conditional distributions for X & Y.

$$X: \begin{cases} P(X=1 \mid Y=-1) = \dfrac{\frac{11}{50}}{\frac{22}{50}} = \dfrac{1}{2} \\ P(X=2 \mid Y=-1) = \dfrac{\frac{8}{50}}{\frac{22}{50}} = \dfrac{8}{22} \\ P(X=3 \mid Y=-1) = \dfrac{\frac{3}{50}}{\frac{22}{50}} = \dfrac{3}{22} \end{cases}$$

$$\begin{cases} P(X=1 \mid Y=1) = \dfrac{\frac{1}{10}}{\frac{28}{50}} = \dfrac{5}{28} \\ P(X=2 \mid Y=1) = \dfrac{13/50}{28/50} = \dfrac{13}{28} \\ P(X=3 \mid Y=1) = \dfrac{1/5}{28/50} = 10/28 \end{cases}$$

$$Y: \begin{cases} P(Y=-1 \mid X=1) = \dfrac{\frac{11}{50}}{16/50} = \dfrac{11}{16} \\ P(Y=1 \mid X=1) = \dfrac{1/10}{16/50} = \dfrac{5}{16} \end{cases}$$

$$\begin{cases} P(Y=-1 \mid X=2) = \dfrac{8/50}{21/50} = 8/21 \\ P(Y=1 \mid X=2) = \dfrac{13/50}{21/50} = 13/21 \end{cases}$$

$$\begin{cases} P(Y=-1 \mid X=3) = \dfrac{3/50}{13/50} = 3/13 \\ P(Y=1 \mid X=3) = \dfrac{1/5}{13/50} = 10/13 \end{cases}$$

No we can calculate the conditional expectations:

$$E(X \mid Y=-1) = 1\cdot\frac{11}{22} + 2\cdot\frac{8}{22} + 3\cdot\frac{3}{22} = \frac{36}{22}$$

$$E(X \mid Y=1) = \frac{61}{28}$$

and similarly

$$E(Y \mid X=1) = -1\,\frac{11}{16} + 1\cdot\frac{5}{16} = -\frac{6}{16}$$

$$E(Y \mid X=2) = \frac{5}{21}$$

$$E(Y \mid X=3) = \frac{7}{13}.$$

Finally

$$A = \left\{ \left(1, -\frac{6}{16}\right), \left(2, \frac{5}{21}\right), \left(3, \frac{7}{13}\right) \right\}$$

$$B = \left\{ \left(\frac{36}{22}, -1\right), \left(\frac{61}{28}, 1\right) \right\}.$$