ERASMUS, B4 Course
Winter 2020 / 2021
FPM&St, December 10, part I

<u>Subject</u>: Limit theorems in P.T.

<u>introduction</u>

We begin with the formulation of the <u>following problem</u>:

<u>Problem 1</u>

Assume that R ~~[illegible]~~ is a set of all distributions (discrete and continuous).

Then each $r \in R$ represents the single distribution,

$(r_n)_{n \geq 1}$ - the <u>sequence</u> of distributions.

More specifically, taking r or $(r_n)_{n \geq 1}$ <u>we mean that</u>:

a) there exists a RE

b) RE has its KPM (Ω, Σ, P)

c) for r we have the R.V. $X: \Omega \to \mathbb{R}$;

for $(r_n)_{n \geq 1}$ we have the sequence of R.V's $(X_n)_{n \geq 1}$, where $X_n: \Omega \to \mathbb{R}$

Let $(X_n)_{n \geq 1}$ be given as above. <u>How</u> to <u>define</u> the <u>convergence</u> of such a sequence?

What is to be its <u>LIMIT?</u>

Before we answer the above questions, let us return to the classical problem of convergence, namely to the theory of real sequences.

Let $(a_n)_{n\geq 1}$, $a_n \in \mathbb{R}$ be given.

By definition we say that $(a_n)_{n\geq 1}$ is convergent if there exits a number $g \in \mathbb{R}$, such that

$$\underset{\varepsilon>0}{\forall} \ \underset{n_0}{\exists} \ \underset{n>n_0}{\forall} \ |a_n - g| < \varepsilon,$$

So $a_n \in (g-\varepsilon, g+\varepsilon)$ for every $n>n_0$.

Then we write $(a_n) \longrightarrow g$, or simply $a_n \longrightarrow g$.

Now our situation is more complicated, because X_n is not a number! On the other hand, if we have $(X_n)_{n\geq 1}$, then for every fixed $\omega_0 \in \Omega$, $(X_n(\omega_0))_{n\geq 1}$ is a sequence of numbers!

In probability theory we consider <u>at least two types of convergence</u> :

① "Almost sure convergence" (a.s.), where by the def. we say that $(X_n)_{n \geq 1}$ is converget a.s. iff $\exists$ r.v. X_0 such that

$$\mathbb{P}(\{\omega \in \Omega : X_n(\omega) \longrightarrow X_0(\omega)\}) = 1,$$

and we write $X_n \xrightarrow{a.s.} X_0$.

② "Convergence by distribution" (d), where by the def. we say that $(X_n)_{n \geq 1}$ is convergent d iff $\forall_{t \in \mathbb{R}}$ $F_{X_n}(t) \longrightarrow F_{X_0}(t)$, and we write $X_n \xrightarrow{d} X_0$.

Our first theorem explains, that in fact we have two different types of convergent, namely

<u>Th</u>. For every sequence r.v.'s (X_n),

$$X_n \xrightarrow{d} X_0 \Longrightarrow X_n \xrightarrow{a.s} X_0$$

but <u>not vice-versa.</u>

<u>Limit Theorems</u>.

(I) The strong LAW of the LARGE NUMBERS of BERNOULLI (SLLNB)

<u>Problem 2</u>.

Let us take X with $\begin{array}{c|c} 0 & 1 \\ \hline q & p \end{array}$.

How to find a value p?

<u>Problem 3</u>.

We know, that for given RE and its p.d. represented by random variable X, the <u>result</u> of the <u>observation</u> of RE is $EX = m$ – the mean value of X.

It is not problem to find EX if we know that distribution. But we cannot to assume that we know the distribution during the observation process. So, how else to determine this number - EX.

Remark

We note, that in P2, $p = EX$.

As we will see in a moment, SLLNB gives the solution of the Problem 2 & Problem ?.

Formulation of SLLNB THEOREM

Assume we observe some R.E. about which we only know, that X has its second moment, so X has EX and $\operatorname{var} X$ (but we do not know the value of EX and $\operatorname{var} X$!). We make the observation in the form of an infinite series of repetitions of X, so we have the sequence $(X_n)_{n \geq 1}$, when

$$\forall_{n \geq 1} \; d(X_n) = d(X).$$

Assume that in addition the members of (X_n) are stochastically independent.

Let us take the event

$$A = \{\omega \in \Omega : \text{the sequence } \underbrace{\frac{1}{n}\left(X_1(\omega) + X_2(\omega) + \dots + X_n(\omega)\right)}_{\text{the average}} \longrightarrow m = EX\}$$

Then (SLLNB says that)

$$P(A) = 1$$

REMARKS

1) The above theorem in the first different (weaker) version was proven first by J. Bernoulli.

2) The proof of the above theorem is too hard, and we ~~on~~ will skip it.

3) Insteadof, we will show below the proof of a weaker version, namely

Let $(X_n)_{n\geq 1}$ be as above, so

- $d(X_n) = d(X)$, $n \geq 1$
- (X_n) are s. ind.
- EX^2 exists.

Then $\forall \varepsilon > 0$

$$(*) \quad P\left(\left\{\omega \in \Omega : \left|\frac{X_1 + \ldots + X_n}{n}(\omega) - m\right| \geq \varepsilon\right\}\right) \to 0$$

Proof

Let $Y_n = \frac{1}{n}(X_1 + \ldots + X_n)$.

Then: $EY_n = \frac{1}{n} \cdot nEX = m$

$$Var(Y_n) = \frac{1}{n^2} n \, Var X = \frac{\sigma^2}{n} \quad (\sigma^2 = var X).$$

It means, that for $\{Y_n\}$ $(n \geq 2)$ we can apply the Tshebyshuff inequality

$$P(\{\omega \in \Omega : |Y_n(\omega) - m| \geq \varepsilon\}) \leq \frac{1}{t^2} = \frac{\sigma_{Y_n}^2}{\varepsilon^2}$$

where $\varepsilon = t\sigma_{Y_n}$ $t > 0$

$$t = \frac{\varepsilon}{\sigma_{Y_n}}$$

But $\sigma_{Y_n}^2 = \frac{\sigma^2}{n}$, $\left(\sigma^2 = var(X)\right)$, so

finally

$$P(\{\omega \in \Omega : |Y_n(\omega) - m| \geq \varepsilon\}) \leq \frac{\sigma^2}{n\varepsilon^2} =$$

$$\frac{\sigma^2}{\varepsilon^2}\frac{1}{n}.$$

But $\frac{1}{n} \to 0$ and $\frac{\sigma^2}{\varepsilon^2}$ is fixed,

so $(*)$ is truth!

4) From SLLNB follows that

$\exists\, \Omega_0 \subset \Omega$, $P(\Omega_0) = 1$, that

for every $\omega_0 \in \Omega_0$

the (real sequence) $(a_n)_{n\geq 1}$, where

$$a_n = \frac{1}{n}\left(X_1(\omega_0) + X_2(\omega_0) + \dots + X_n(\omega_0)\right), \quad n \geq 1$$

is convergent, and the limit of its is equal to $m = EX$, so

$$a_n \longrightarrow m.$$

It means, that for given (small) $\varepsilon > 0$ the condition

$|a_n - m| < \varepsilon$ is satisfied for all $n \geq n_0$, for some n_0.

So, if n is suffiient large, then a_n is very close to m, so we can write (estimation rule!)

$$a_n \approx m.$$

5). Let us return to the Problem 2.

$$X : \begin{array}{c|c} 0 & 1 \\ \hline q & p \end{array}$$

Let $(X_n)_{n\geq 1}$ be such that

$d(X_n) = d(X)$, $n \geq 1$ and s.i.

We note, that for every $\omega \in \Omega$,

$$\frac{1}{n}(X_1(\omega) + X_2(\omega) + \dots + X_n(\omega)) = \frac{\# \text{ of the } j \ X_j(\omega) = 1}{\# \text{ of repetitions}}$$

For this reason, the above number is denoted by $f_n(\omega)$ and is called the FREQUENCE.

So from SLLNB, for X as above we have:

with probability 1,

$$f_n(\omega) \longrightarrow p \quad (p = EX)$$

Thus we can approximate p by $f_n(\omega)$!

$f_n(\omega) \simeq p$ with probability 1,

Therefore: p is called the theoretical probability,

$f_n(\omega)$ - the empirical probability.

6) Let us take an example.

Suppose that for every $n \geq 2$ and fixed $p \in (0,1)$

$$B_n \in B(n,p) \text{ (Bernoulli distribution)}$$

From factorization property of $B(n,p)$ we know that we have

$$B_n(\omega) = X_{n,1}(\omega) + X_{n,2}(\omega) + \dots + X_{n,n}(\omega),$$

where $\forall_{1 \leq k \leq n}$ $X_{n,k} : \begin{array}{c|c} 0 & 1 \\ \hline q & p \end{array}$ and

are s.i.

For $n \geq 2$ and fixed $\varepsilon > 0$ we consider the event

$$A_{\varepsilon,n} = \{\omega \in \Omega : |\tfrac{1}{n} B_n(\omega) - p| \geq \varepsilon\}$$

$$\tfrac{1}{n} B_n(\omega) = \tfrac{1}{n}(X_{1,n}(\omega) + \dots + X_{n,n}(\omega))$$

We are going to approximate $P(A_{\varepsilon,n})$.

Since: $E(\frac{1}{n} B_n) = \frac{1}{n} \cdot np = p$

$$\operatorname{var}(\tfrac{1}{n} B_n) = \frac{1}{n^2} \cdot npq = \frac{pq}{n},$$

if we put $\varepsilon = t\sigma_n = t\sqrt{\operatorname{var}\frac{1}{n}B_n} = t\sqrt{\frac{pq}{n}}$,

by Tchebychev inequality we obtain

$$P(\{\omega \in \Omega : |\tfrac{1}{n} B_n(\omega) - p| \geq \underset{t\sigma_n}{\varepsilon}\}) \leq \left(\frac{\varepsilon^2}{\sigma_n^2}\right)^{-1} = \frac{pq}{n\varepsilon^2}$$

or equivalently

(**) $P(A^c_{\varepsilon,n}) > 1 - \dfrac{pq}{n\varepsilon^2}$

Let us do a numeric simulation:

$$p = q = 0{,}5\,, \quad \varepsilon = 10^{-k}\,, \quad k \geq 2\,.$$

Then

$$\frac{pq}{n\varepsilon^2} = \frac{1/4}{n\,10^{-2k}} = \frac{1}{4\cdot n\cdot 10^{-2k}}\,.$$

Assume that for $k \geq 2$ we have

$$1 - \frac{1}{4\cdot n\cdot 10^{-2k}} \geq 1 - 10^{-3} \iff$$

$$\frac{1}{4\cdot n\cdot 10^{-2k}} \leq 10^{-3} \iff$$

$$n > \frac{1}{4}\cdot 10^{3+2k}\,.$$

We note, that only for $k = 2$ we have

(xx) is satisfied for

$$n > \frac{1}{4}\cdot 10^{7} = \underline{2.500.000}\,,$$ so

$$P\left(\{\omega\in\Omega : \left|\frac{1}{n}B_n(\omega) - p\right| < 0{,}01\}\right) \geq 0{,}99,$$

if $n \geq 2.500.000$.

I think you understood why in the name of Bernoulli th. we have the words LARGE NUMBERS!

//.