

Subject: A concept of the joint distribution of the random vector

Introduction

So far we assumed, that whenever we were observing some RE, we only need one random variable.

Now we are going to use more. First we assume that for given (Ω, Σ, P) we have two random variables, namely

$$X, Y: \Omega \rightarrow \mathbb{R} \text{ and}$$

$$\forall t, s \in \mathbb{R}, \{ \omega \in \Omega : X(\omega) < t \}, \{ \omega \in \Omega : Y(\omega) < s \} \in \Sigma.$$

It means, that the result of our observation is as follows

$$\Omega + \omega \rightarrow (X, Y)(\omega) = (X(\omega), Y(\omega)) \in \mathbb{R} \times \mathbb{R}$$

The object (X, Y) is called a random vector of rank 2 with coordinates: X , and Y respectively.

And generally, for $n \geq 2$, X_1, X_2, \dots, X_n random variables by the random vector of rank n we understand the object (X_1, X_2, \dots, X_n) .

Then

$$\Omega \ni \omega \longrightarrow (X_1, X_2, \dots, X_n)(\omega) = (X_1(\omega), X_2(\omega), \dots, X_n(\omega))$$

and X_j is called the j^{th} coordinate $\in \mathbb{R}^m$

Assumption.

In this course we will only deal with 2-rank vectors
variables (X, Y) .

Def 1

By the joint distribution of the v.v. (X, Y)
we understand the multifunction

$$\Omega \times \mathbb{R}^2 \ni (t, s) \longrightarrow F_{(X, Y)}(t, s) \stackrel{\text{def}}{=} P\left\{\{\omega \in \Omega : X(\omega) < t \text{ and } Y(\omega) < s\}\right\}$$

Then $F_{(X, Y)}$ is called the cumulative probability function

Remarks.

$$1^{\text{o}} \text{ let } A_t = \{\omega \in \Omega : X(\omega) < t\}, \quad t, s \in \mathbb{R}$$

$$B_s = \{\omega \in \Omega : Y(\omega) < s\}.$$

then $F_{(X, Y)}(t, s) = P(A_t \cap B_s)$. So if in particular
 X & Y are statistically independent, then

$$F_{(X,Y)}(t,s) = P(A_t \cap B_s) = F_X(t) F_Y(s)$$

and we can say that the joint distribution is equal to the product of distributions.

2^o Let's take $F_{(X,Y)}$ for (X,Y) , and fix t_0, s_0 .

Then we can consider respectively:

$$\mathbb{R} \rightarrow t \longrightarrow F_{(X,Y)}(t, s_0)$$

$$\mathbb{R} \rightarrow s \longrightarrow F_{(X,Y)}(t_0, s)$$

Now we have

$$F_{(X,Y)}(t, s_0) = P(\{v \in \mathbb{R}: X(v) < t \wedge Y(v) < s_0\})$$

$$\xrightarrow{s_0 \rightarrow +\infty} P(\{v \in \mathbb{R}: X(v) < t \wedge Y(v) \in \mathbb{R}\}) =$$

$$= P(\{v \in \mathbb{R}: X(v) < t\} \cap \mathbb{R}) = P(\{v \in \mathbb{R}: X(v) < t\}) = F_X(t).$$

Similarly,

$$F_{(X,Y)}(t_0, s) \xrightarrow{t_0 \rightarrow +\infty} F_Y(s), \quad t, s \in \mathbb{R}.$$

This gives us the following theorem

Th1. (on marginal distribution)

For any r.v. (X, Y) and C.P.F. $F_{(X,Y)}$,

$$F_X(t) = \lim_{s \rightarrow +\infty} F_{(X,Y)}(t, s), \quad t \in \mathbb{R}$$

$$F_Y(s) = \lim_{t \rightarrow +\infty} F_{(X,Y)}(t, s), \quad s \in \mathbb{R}.$$

So F_X and F_Y are determined by $F_{(X,Y)}$.

If in addition X and Y are s.i., then

$$F_{(X,Y)} = F_X F_Y.$$

Remark. In general, $F_{(X,Y)} \neq F_X F_Y$ \neq .

The case of discrete r.v.

We will limit our considerations further to the case where X and Y are discrete and finite, so we have:

$$X(\Omega) = \{x_1, x_2, \dots, x_n\}, n \geq 2$$

$$Y(\Omega) = \{y_1, y_2, \dots, y_m\}, m \geq 2$$

$\forall i \in \{1, \dots, n\} \quad p_i := P(\Omega \cap \{X(\omega) = x_i\})$,

$\forall j \in \{1, \dots, m\} \quad q_j := P(\Omega \cap \{Y(\omega) = y_j\})$, and

$$\text{of course } \sum_{i=1}^n p_i = \sum_{j=1}^m q_j = 1.$$

Then we have

$$(X, Y)(\Omega) = \{(x_i, y_j), \quad i=1, \dots, n, \quad j=1, \dots, m\}$$

For every $i \in \{1, \dots, n\}, \quad j \in \{1, \dots, m\}$ we define

$$A_{ij} = \{\omega \in \Omega : (X, Y)(\omega) = (x_i, y_j)\}.$$

Let $p_{ij} = P(A_{ij})$ and $\mathcal{A} = \{A_{ij}, \quad i=1, \dots, n, \quad j=1, \dots, m\}$.

Proposition 1.

For discrete finite vector (X, Y) as above we have

1. The family \mathcal{A} is a partition of Ω

2. $\forall i, j \quad p_{ij} \in (0, 1)$

$$3. \quad \sum_j \sum_i p_{ij} = 1$$

$$\text{h} \quad \forall \quad \sum_{j=1}^m p_{ij} = p_i$$

$$\forall \quad \sum_{i=1}^n p_{ij} = q_j$$

Proof. It is a task for you.

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Now let's define the matrix $P \in M_{n \times m}$, where

$$P = [p_{ij}]_{n \times m} = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1m} \\ p_{21} & p_{22} & \dots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \dots & p_{nm} \end{bmatrix}$$

By Prop 1 the sum of all elements from the ith rows of P $\rightarrow \sum_{j=1}^m p_{ij}$ defining the distribution of X , for $1 \leq i \leq n$

and similarly, the sum of all elements for the j^{th} columns, $\sum_{i=1}^n p_{ij}$ define the distribution of Y for $1 \leq j \leq m$.

Further, we denote

$$P_{i.} \stackrel{\text{def}}{=} \sum_{j=1}^m P_{ij} (= p_i)$$

$$P_{.j} \stackrel{\text{def}}{=} \sum_{i=1}^n P_{ij} (= q_j).$$

Th2 (on the distribution of finitely discrete r.v.)

Every finite discrete r.v. (X, Y) can be uniquely represented by the matrix $P = [P_{ij}]_{n \times m}$, where

(i) $n = |X(\Omega)|$, $m = |Y(\Omega)|$

(ii) $P_{ij} = P\{ \text{Event: } (X, Y)(\omega) = (x_i, y_j) \},$

where $X(\Omega) = \{x_1, \dots, x_n\}$

$Y(\Omega) = \{y_1, \dots, y_m\},$

(iii) $P_{i.}$, $i=1, \dots, n$ and $P_{.j}$, $j=1, \dots, m$ determine the distributions of X & Y , so are margins of (X, Y) .

In particular, X and Y are stoch. ind. iff

$$\forall_{i,j} \quad P_{i.} P_{.j} = P_{ij} \quad \iff$$

And we verify, whenever we have the matrix
 $[P_{ij}]_{n \times m}$ with the properties (iii), the
 there exists at least one RE and two
 r.v. X, Y for which (i) and (ii) hold.

Remark

The matrix P is called the standard representation
 of the r.v. (X, Y) , we write
 $(X, Y) \rightarrow P_{(X, Y)}$.

Example 1

Prove H.A for some $a, b, c \in \mathbb{N}_2$, the matrix

$$A = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{12}, \frac{1}{12} & \frac{1}{12} \\ a, b & c \end{bmatrix} \text{ represents the joint}$$

distribution for some r.v. (X, Y) .

We are going to apply Th. 2.

The a, b, c must satisfy the following conditions:

$$1) \quad a, b, c \in (0, 1)$$

$$2) \quad 3 \cdot \frac{1}{12} + 3 \cdot \frac{1}{12} + (a+b+c) = 1$$

$$3) \left(\frac{1}{6} + \frac{1}{12} + a \right) + \left(\frac{1}{6} + \frac{1}{12} + b \right) + \left(\frac{1}{6} + \frac{1}{12} + c \right) = 1$$

It means that

$$(*) \quad a+b+c = \frac{1}{4} \quad \text{and} \quad a, b, c \in (0, 1)$$

It is (almost) clear that there are infinitely many solutions of (*),

let a_0, b_0, c_0 be one of them. Then

the marginal distributions are the following

$$X: \quad P_{1,0} = \frac{1}{12}, \quad P_{2,0} = \frac{1}{12}, \quad P_{3,0} = a_0 + b_0 + c_0 = \frac{1}{4}$$

$$Y: \quad P_{\cdot,1} = a_0 + \frac{1}{4}, \quad P_{\cdot,2} = b_0 + \frac{1}{4}, \quad P_{\cdot,3} = c_0 + \frac{1}{4}.$$

TASK 1

When X and Y from the ex. 1 are st. ind.?

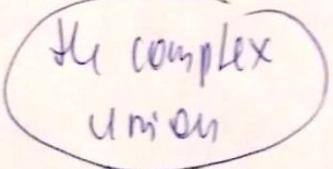
The problem of distribution of $X+Y$ - as an application of the joint distribution.

We will show how to establish the distribution of $Z = X+Y$ if the joint distribution is known.

By assumption we have the matrix $P = [P_{ij}]_{m \times n}$

For $Z = X+Y$ we can write

$$Z(\Omega) = X(\Omega) \oplus Y(\Omega) \stackrel{dt}{=} \left\{ c \in \mathbb{R} : c = a + b, a \in X(\Omega), b \in Y(\Omega) \right\}$$



Then Z has a discrete distribution, and for $c \in Z(\Omega)$ we have

$$P(\{\omega \in \Omega : Z(\omega) = c\}) = P(\{\omega \in \Omega : \exists (X(\omega), Y(\omega)) = (a, b) \text{ s.t. } c = a + b\})$$

For given $a \in X(\Omega)$ and $b \in Y(\Omega)$, let

$$A_{a,b} = \{ \omega \in \Omega : (X(\omega), Y(\omega)) = (a, b) \}$$

Then by addition property of P ,

$$(\#) P(\{v \in V : Z(v)=c\}) = \sum_{a+b=c} P(A_{a,b}) = \sum_{a+b=c} p_{ab}$$

In particular, if X and Y are s.i., then

$$p_{ab} = p_a \cdot p_{,b} \text{, and hence}$$

$$(\#\#) P(\{v \in V : Z(v)=c\}) = \sum_{a+b=c} p_a \cdot p_{,b}.$$

The operations (#) & (##) are called the convolution operation of the distribution.

Example 2.

Let $X(v) = \{1, 2, 3\}$, $Y(v) = \{-1, 0, 1\}$ and

$$(X, Y) \rightarrow P = \begin{matrix} & \begin{matrix} Y=-1 & Y=0 & Y=1 \end{matrix} \\ \begin{matrix} X=1 \\ X=2 \end{matrix} & \begin{bmatrix} 1/6 & 1/6 & 1/3 \\ 1/3 & 1/3 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix} \end{matrix}$$

$$P(\{v \in V : X(v)=1\}) = 1/3, \quad P(\{v \in V : X(v)=2\}) = 1/2$$

$$P(\{v \in V : Y(v)=-1\}) = P(\{v \in V : Y(v)=1\}) = 1/2$$

So, we can write

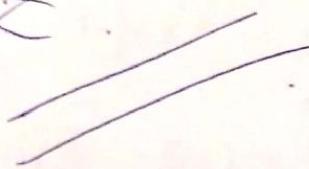
$$X: \begin{array}{c|c} 1 & 2 \\ \hline 1_1 & e_{11} \end{array} \quad Y: \begin{array}{c|c} 1 & 1 \\ \hline 1_2 & 1_2 \end{array}$$

} as marginally

We make X and Y an independent (WMV?)

and $Z(\eta) = \{0, 1, 2, 3\}$.

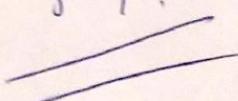
Please apply (#H) for Z .



TASK 2.

When $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represents some n.v (X, Y) .

When X and Y are s.i.



TASK 3

Compute $Z = X + Y$ for X, Y from TASK 2.

