ERASMUS, B4 Course
Winter semester 2020/2021, December 17

---

Subject: A concept of the joint distribution of the random vector

Introduction

So far we assumed, that whenever we were observing some RE, we only need one random variable.

Now we are going to use more. First we assume that for given $(\Omega, \Sigma, P)$ we have two random variables, namely

$$X, Y: \Omega \longrightarrow \mathbb{R} \text{ and}$$

$$\forall t, s \in \mathbb{R}, \{\omega \in \Omega: X(\omega) < t\}, \{\omega \in \Omega: Y(\omega) < s\} \in \Sigma.$$

It means, that the result of our observation is as follows

$$\Omega \ni \omega \longrightarrow (X, Y)(\omega) = (X(\omega), Y(\omega)) \in \mathbb{R} \times \mathbb{R}$$

The object $(X, Y)$ is called a random vector of rank 2 with coordinates: $X$, and $Y$ respectively.

And generally, for $n \geq 2$, $X_1, X_2, \ldots, X_n$ random variables by the random vector of rank $n$ we understand the object $(X_1, X_2, \ldots, X_n)$.

1

Then

$$\Omega \ni \omega \longrightarrow (X_1, X_2, .., X_n)(\omega) = (X_1(\omega), X_2(\omega), .., X_n(\omega))$$

and $X_j$ is called the $j^{th}$ coordinate $\in \mathbb{R}^n$

Assumption.

In this course we will only deal with 2-random vectors variables $(X, Y)$.

Def 1

By the joint distribution of the v.v. $(X, Y)$ we understand the multi function

$$\mathbb{R} \times \mathbb{R} \ni (t, s) \longrightarrow F_{(X,Y)}(t,s) \stackrel{df}{=} P(\{\omega \in \Omega : X(\omega) < t \,\&\, Y(\omega) < s\})$$

Then $F_{(X,Y)}$ is called the cummulative probability function

Remarks.

1° Let $A_t = \{\omega \in \Omega : X(\omega) < t\}$, $\quad t, s \in \mathbb{R}$

$B_s = \{\omega \in \Omega : Y(\omega) < s\}$.

Then $F_{(X,Y)}(t,s) = P(A_t \cap B_s)$. So if in particular $X$ & $Y$ are stochastically independent, then

$$F_{(X,Y)}(t,s) = P(A_t)P(B_s) = F_X(t)F_Y(s)$$

and we can say that the joint distribution is equal to the product of distributions.

2° Let's take $F_{(X,Y)}$ for $(X,Y)$, and fix $t_0, s_0$.

Then we can consider respectively:

$$\mathbb{R} \ni t \longrightarrow F_{(X,Y)}(t, s_0)$$

$$\mathbb{R} \ni s \longrightarrow F_{(X,Y)}(t_0, s).$$

Now we have

$$F_{(X,Y)}(t, s_0) = P(\{\omega\in\Omega : X(\omega)<t \wedge Y(\omega)<s_0\})$$

$$\xrightarrow[s_0\to+\infty]{} P(\{\omega\in\Omega : X(\omega)<t \wedge Y(\omega)\in\mathbb{R}\} =$$

$$= P(\{\omega\in\Omega: X(\omega)<t\}\cap\Omega\}) = P(\{\omega\in\Omega : X(\omega)<t\}) = F_X(t).$$

Similarly,

$$F_{(X,Y)}(t_0, s) \xrightarrow[t_0\to+\infty]{} F_Y(s), \quad t, s\in\mathbb{R}.$$

$\supset$

This gives us the following theorem

<u>Th 1</u>. (On marginal distributions)

For any r.v. $(X,Y)$ and C.P.F. $F_{(X,Y)}$,

$$F_X(t) = \lim_{s \to +\infty} F_{(X,Y)}(t,s), \quad t \in \mathbb{R}$$

$$F_Y(s) = \lim_{t \to +\infty} F_{(X,Y)}(t,s), \quad s \in \mathbb{R}.$$

So $F_X$ and $F_Y$ are determined by $F_{(X,Y)}$.

If in addition $X$ and $Y$ are <u>s.i.</u>, then

$$F_{(X,Y)} = F_X F_Y .$$

<u>Remark</u> In general, $F_{(X,Y)} \neq F_X F_Y$.

<u>The case of discrete r.v.</u>

We will limit our considerations further to the case where $X$ and $Y$ are <u>discrete and finite</u>, so we have:

$X(\Omega) = \{x_1, x_2, \ldots, x_n\}$ $n \geq 2$

$Y(\Omega) = \{y_1, y_2, \ldots, y_m\}$, $m \geq 2$

$\forall_{1 \le i \le n} \; p_i := P(\{\omega \in \Omega : X(\omega) = x_i\})$,

$\forall_{1 \le j \le m} \; q_j := P(\{\omega \in \Omega : Y(\omega) = y_j\})$, and

of course $\sum_{1 \le i \le n} p_i = \sum_{1 \le j \le m} q_j = 1$.

Then we have

$(X,Y)(\Omega) = \{(x_i, y_j), \; i = 1 \ldots, n, \; j = 1 \ldots, m\}$

For every $i \in \{1 \ldots n\}$, $j \in \{1 \ldots m\}$ we define

$A_{ij} = \{\omega \in \Omega : (X,Y)(\omega) = (x_i, y_j)\}$.

Let $p_{ij} = P(A_{ij})$ and $\mathcal{A} = \{A_{ij}, \; \substack{i = 1 \ldots n \\ j = 1 \ldots m}\}$.

Proposition 1.

For discrete finite vector $(X,Y)$ as above we have

1. The family $\mathcal{A}$ is a partition of $\Omega$
2. $\forall i,j \; p_{ij} \in (0,1)$
3. $\sum_j \sum_i p_{ij} = 1$

ii' $\forall_{1 \leq i \leq n} \sum_{j=1}^{m} P_{ij} = P_i$

$\forall_{1 \leq j \leq m} \sum_{i=1}^{n} P_{ij} = q_j$

Proof. It is a task for YOU.

//

Now let's define the matrix $\mathbb{P} \in M_{n \times m}$, where

$$\mathbb{P} = [P_{ij}]_{n \times m} = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1m} \\ P_{21} & P_{22} & \dots & P_{2m} \\ \hline P_{i1} & P_{i2} & \dots & P_{im} \\ P_{n1} & P_{n2} & \dots & P_{nm} \end{bmatrix}$$

By Prop 1 the sum of all elements from the $i$th rows of $\mathbb{P}$, $\sum_{j=1}^{m} P_{ij}$ defining the distribution of X, for $1 \leq i \leq n$,

and similarly, the sum of all elements for the $j^{th}$ columns, $\sum_{i=1}^{n} P_{ij}$ define the distribution of Y for $1 \leq j \leq m$.

Further, we denote

$$P_{i\cdot} \stackrel{df}{=} \sum_{j=1}^{m} P_{ij} \; (= P_i)$$

$$P_{\cdot j} \stackrel{df}{=} \sum_{i=1}^{n} P_{ij} \; (= q_j).$$

<u>Th2 (on the distribution of finite discrete r.v.)</u>

Every finite discrete r.v $(X,Y)$ can be <u>uniquely represented</u> by the matrix $\mathbb{P} = [P_{ij}]_{n\times m}$, where

(i) $n = |X(\Omega)|$, $m = |Y(\Omega)|$

(ii) $P_{ij} = P(\{\omega\in\Omega : (X,Y)(\omega) = (x_i, y_j)\})$,

where $X(\Omega) = \{x_1, \dots, x_n\}$

$Y(\Omega) = \{y_1, \dots, y_m\}$,

(iii) $P_{i\cdot}$, $i=1\dots n$ and $P_{\cdot j}$, $j = 1\dots m$ determine the distributions of $X$ & $Y$, so are marginals to $(X,Y)$.

In particular, $X$ and $Y$ are <u>stoch. ind.</u> iff

$$\forall_{i,j} \; P_{i\cdot} P_{\cdot j} = P_{ij}$$

And vice versa, whenever we have the matrix $[p_{ij}]_{n\times m}$ with the properties (iii), then there exists at least one RE and two r.v. $X, Y$ for which (i) and (ii) hold.

Remark.

The matrix $\mathbb{P}$ is called the ~~stoch~~ stochastic representation of the r.v $(X,Y)$, so we write

$$(X,Y) \longrightarrow \mathbb{P}_{(X,Y)}.$$

Example 1.

Prove that for some $a, b, c \in \mathbb{R}$, the matrix

$$A = \begin{bmatrix} 1/6 & 1/6 & 1/6 \\ 1/12 , & 1/12 & 1/12 \\ a , & b & c \end{bmatrix}$$ represents the joint

distribution for some r.v $(X,Y)$.

We are going to apply Th. 2.

Then $a, b, c$ must satisfy the following conditions:

1) $a, b, c \in (0,1)$

2) $3 \cdot \frac{1}{6} + 3 \cdot \frac{1}{12} + (a+b+c) = 1$

3) $\left(\frac{1}{6} + \frac{1}{12} + a\right) + \left(\frac{1}{6} + \frac{1}{12} + b\right) + \left(\frac{1}{6} + \frac{1}{12} + c\right) = 1$

It means that

(*) $a+b+c = \frac{1}{4}$ and $a, b, c \in (0,1)$

It is (almost) clear that there exists an infinitely many solutions of (*).

Let $a_0, b_0, c_0$ be one of them. Then the marginal distributions are the following

$X: \; P_{1\cdot} = \frac{1}{2}, \; P_{2\cdot} = \frac{1}{4}, \; P_{3\cdot} = a_0 + b_0 + c_0 = \frac{1}{4}$

$Y: \; P_{\cdot 1} = a_0 + \frac{1}{4}, \; P_{\cdot 2} = b + \frac{1}{4}, \; P_{\cdot 3} = c_0 + \frac{1}{4}.$

TASK 1

When $X$ and $Y$ from the ex 1 are st. ind.?

The problem of distribution of $X+Y$ - as an application of the joint distribution.

We will show how to establish the distribution of $Z = X+Y$ if the joint distribution is known.

By assumption we have the matrix $P = [p_{ij}]_{n \times m}$

For $Z = X+Y$ we can write

$$Z(\Omega) = X(\Omega) \oplus Y(\Omega) \overset{df}{=} \{ c \in \mathbb{R} : c = a+b, \; a \in X(\Omega), \; b \in Y(\Omega) \}$$

the complex union

Therefore $Z$ has a discrete distribution, and for $c \in Z(\Omega)$ we have

$$P(\{\omega \in \Omega : Z(\omega) = c\}) = P(\{\omega \in \Omega : \exists_{a,b} \, (X(\omega), Y(\omega)) = (a,b) \; \& \; c = a+b \})$$

For given $a \in X(\Omega)$ and $b \in Y(\Omega)$, let

$$A_{a,b} = \{ \omega \in \Omega : (X(\omega), Y(\omega)) = (a,b) \}$$

Then by additivity property of $\mathbb{P}$,

$$(\#)\ \mathbb{P}(\{\omega\in\Omega : Z(\omega)=c\}) = \sum_{a+b=c} \mathbb{P}(A_{a,b}) = \sum_{a+b=c} p_{ab}$$

In particular, if X and Y are <u>s.i.</u>, then

$p_{ab} = p_{a\cdot}\, p_{\cdot b}$, and therefore

$$(\#\#)\ \mathbb{P}(\{\omega\in\Omega : Z(\omega)=c\}) = \sum_{a+b=c} p_{a\cdot}\, p_{\cdot b}.$$

The operations $(\#)$ & $(\#\#)$ are called the <u>convolution operation</u> of the distributions.

<u>Example 2</u>.

Let $X(\Omega)=\{1,2\}$, $Y(\Omega)=\{-1,1\}$ and

$$(X,Y) \longrightarrow P = \begin{array}{c|cc|c} & Y=-1 & Y=1 & \\ X=1 & 1/6 & 1/6 & 1/3 \\ X=2 & 1/3 & 1/3 & 2/3 \\ \hline & 1/2 & 1/2 & \end{array}, \text{ so}$$

$\mathbb{P}(\{\omega\in\Omega : X(\omega)=1\}) = 1/3$, $\mathbb{P}(\{\omega\in\Omega : X(\omega)=2\}) = 2/3$

$\mathbb{P}(\{\omega\in\Omega : Y(\omega)=-1\}) = \mathbb{P}(\{\omega\in\Omega : Y(\omega)=1\}) = 1/2$

So, we can write

$$\left.\begin{array}{l} X: \begin{array}{c|c} 1 & 2 \\ \hline 1/3 & 2/3 \end{array} \\[2em] Y: \begin{array}{c|c} -1 & 1 \\ \hline 1/2 & 1/2 \end{array} \end{array}\right\} \text{ as marginals}$$

We note that X and Y are independent (WHY?)

and $Z(\Omega) = \{0, 1, 2, 3\}$.

Please apply (##) for $Z$.

TASK 2

When $\mathbb{P} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represents some r.v $(X, Y)$.

When $X$ and $Y$ are s.i.?

TASK 3

Compute $Z = X + Y$ for $X, Y$ from TASK 2.