ERASMUS Bd course

Winter 2020/2021

FPM&St November 19

Subject 1°. The concept of P.D. – continuous case – examples

2°. The parameters of P.D.

We begin with the following example

Ex1. Let $X \in U([0,1])$ and $\lambda > 0$.

We define $Y = -\frac{1}{\lambda}\ln(1-X)$. We will find the distribution of Y.

By assumption there exists at least one RE and its KPM (Ω, Σ, P), such that

$F_X(t) = P(\{\omega \in \Omega : X(\omega) < t\})$ and

$Y(\omega) = -\frac{1}{\lambda}\ln(1-X(\omega))$. Since $X(\omega) \in (0,1)$

the last definition is correct.

Moreover, $\forall \omega \in \Omega$ $1-X(\omega) \in (0,1)$, so

$\ln(1-X(\omega)) < 0$, and consequently $Y(\omega) > 0$.

It means, that for $t \le 0$

$F_Y(t) = 0$.

Therefore, to find F_Y it suffices to assume that $t>0$.
For $t>0$ we have

$$F_Y(t) = P(\{\omega\in\Omega: Y(\omega)<t\}) = P(\{\omega\in\Omega: -\frac{1}{\lambda}\ln(1-X(\omega)) < t\})$$

$$= P(\{\omega\in\Omega: \ln(1-X(\omega)) > -\lambda t\}) =$$

$$= P(\{\omega\in\Omega: 1-X(\omega) > e^{-\lambda t}\}) =$$

$$= P(\{\omega\in\Omega: X(\omega) < 1-e^{-\lambda t}\}) = F_X(1-e^{-\lambda t}).$$

So,

$$F_Y(t) = \begin{cases} 0, & t\le 0 \\ F_X(1-e^{-\lambda t}), & t>0 \end{cases}, \text{ and}$$

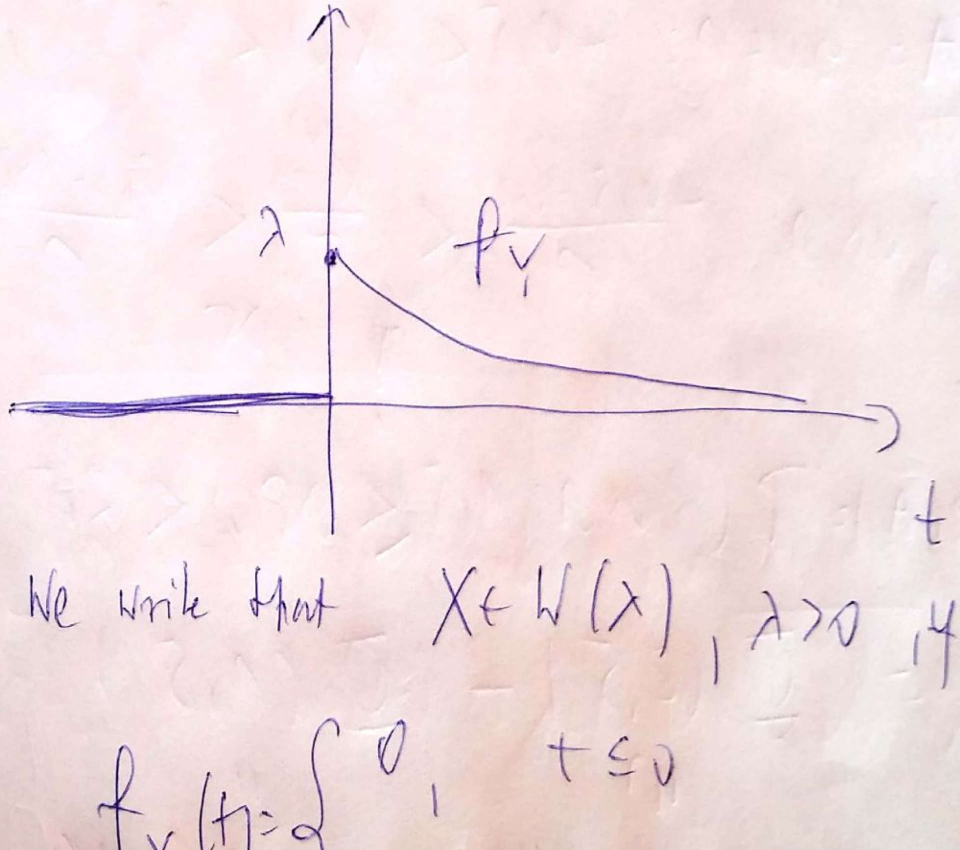
$$f_Y(t) = \begin{cases} 0, & t\le 0 \\ \frac{d}{dt} F_X(1-e^{-\lambda t}), & t>0 \end{cases} =$$

$$= \begin{cases} 0, & t\le 0 \\ f_X(1-e^{-\lambda t})e^{-\lambda t}\cdot\lambda, & t>0 \end{cases}$$

But
$$f_X(u)=\begin{cases}1, & u\in[0,1]\\ 0, & u\notin[0,1]\end{cases}$$

and
$$u=1-e^{-\lambda t}\in[0,1] \Leftrightarrow t>0,$$

so finally
$$f_Y(t)=\begin{cases}0, & t\le 0\\ \lambda e^{-\lambda t}, & t>0\end{cases}$$



Def 1 We write that $X\in W(\lambda)$, $\lambda>0$, if

$$f_X(t)=\begin{cases}0, & t\le 0\\ \lambda e^{-\lambda t}, & t>0\end{cases}$$

X is called an exponential distribution with param. λ

No ve find the F_X for $X \in W(\lambda)$.

From general theory

$$F_X(t) = \int_{-\infty}^{t} f_X(u)\,du = \begin{cases} 0 & t \leq 0 \\ \int_{-\infty}^{t} \overset{f_X(u)}{\cancel{\lambda e^{-\lambda u}}}\,du, & t > 0 \end{cases}$$

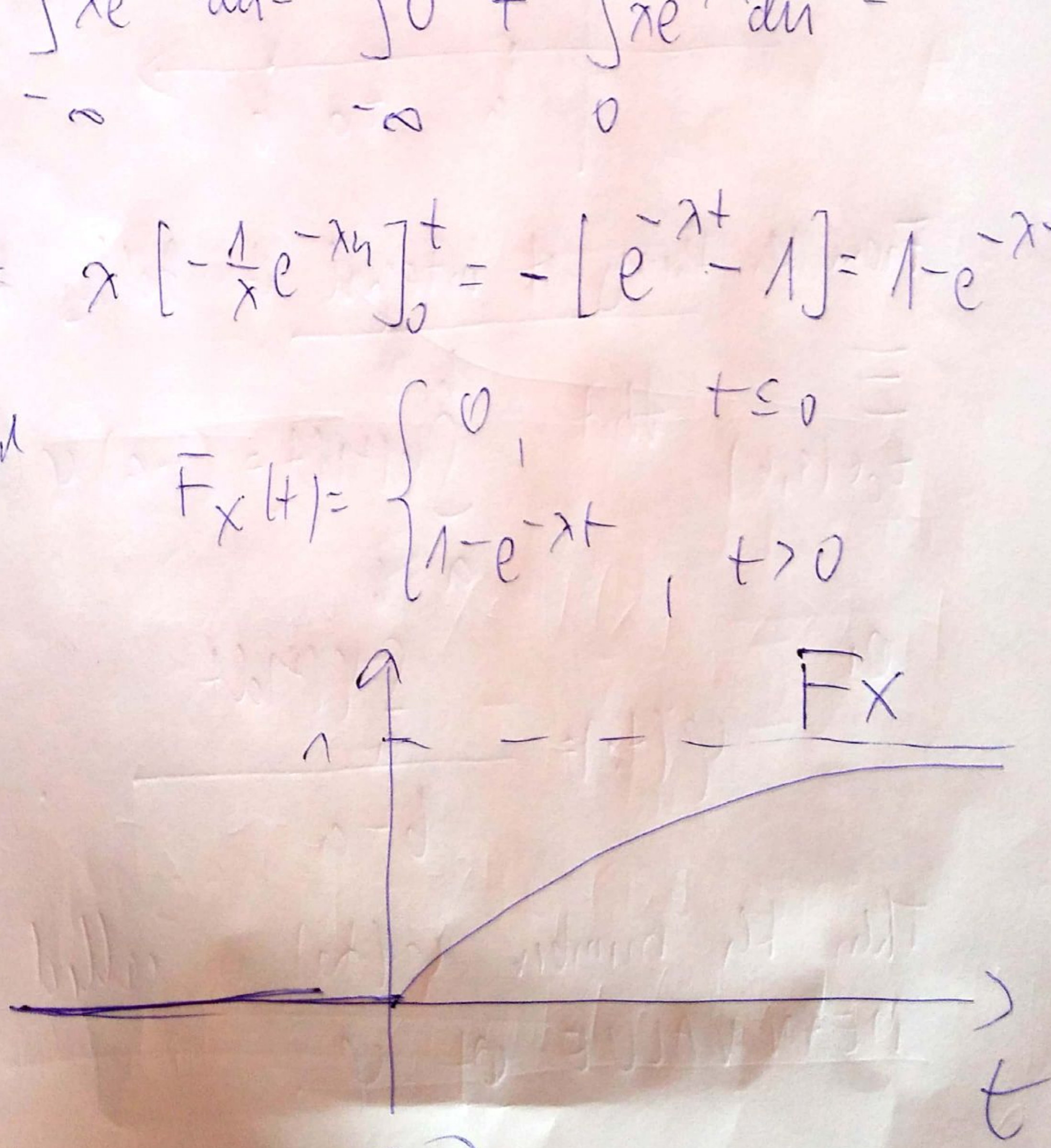
Bu A

$$\int_{-\infty}^{t} \lambda e^{-\lambda u}\,du = \int_{-\infty}^{0} 0 + \int_{0}^{t} \lambda e^{-\lambda u}\,du =$$

$$= \lambda \left[-\frac{1}{\lambda} e^{-\lambda u} \right]_0^t = -\left[e^{-\lambda t} - 1 \right] = 1 - e^{-\lambda t}$$

and

$$F_X(t) = \begin{cases} 0, & t \leq 0 \\ 1 - e^{-\lambda t}, & t > 0 \end{cases}$$



Gaussian distribution

We say that $X \in N(m, \sigma^2)$, where $m \in \mathbb{R}$, $\sigma > 0$ if the density function f_X is given as follows

$$f_X(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-m)^2}{2\sigma^2}}, \quad t \in \mathbb{R},$$

where „e" denotes the Euler's number.

~~PA~~ STATEMENT.

Let $X \in N(m, \sigma^2)$. Then

$$Y = \frac{X-m}{\sigma} \in N(0, 1) \ (\text{so } m=0, \sigma=1).$$

Proof.

$$F_Y(t) = P(\{\omega \in \Omega : Y(\omega) < t\}) = P(\{\omega \in \Omega : \frac{X(\omega)-m}{\sigma} < t\})$$

$$= P(\{\omega \in \Omega : X(\omega) < m + t\sigma\}) = F_X(m + t\sigma).$$

Hence

$$f_Y(t) = \frac{d}{dt} F_Y(t) = F_X'(m + t\sigma) \cdot (m + t\sigma)' =$$

$$= f_X(m + t\sigma)\cdot \sigma .$$

Therefore by def. of f_X we have

$$f_Y(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(m+t\sigma-m)^2}{2\sigma^2}} \cdot \sigma =$$

$$= \frac{1}{\sqrt{2\pi}} e^{-t^2/2} = f_X(t)\Big|_{\substack{m=0\\ \sigma=1}}$$

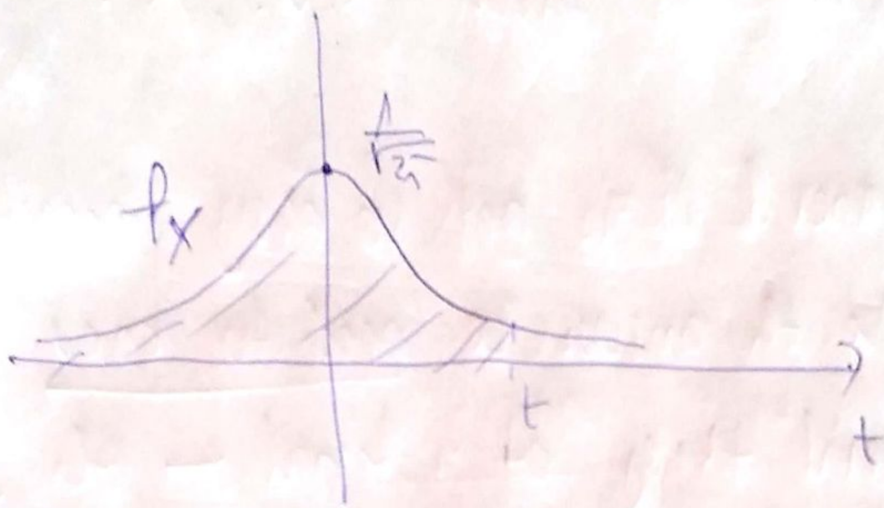
and $Y \in N(0,1)$. //

If $X \in N(0,1)$ we say that we have a <u>standard gaussian distribution</u>. Then the C.P.D.F. we denote by Φ.

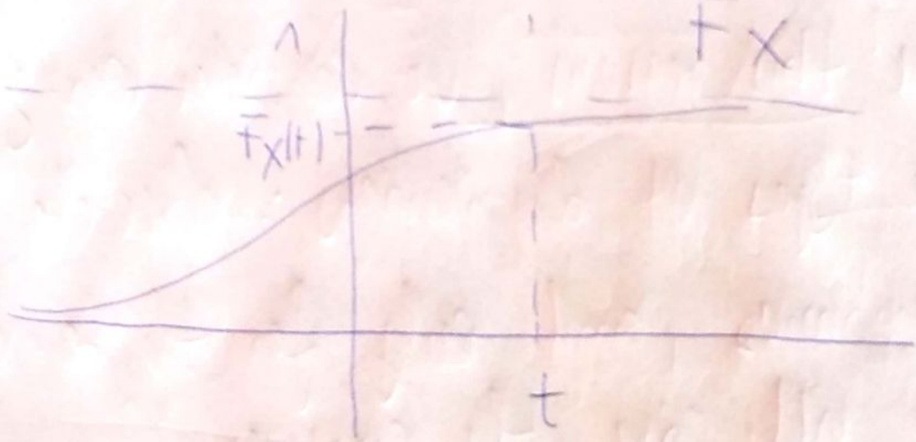
<u>Remarks concerning $N(0,1)$</u>

1). The procedure realized in the STATMENT is called „standarization procedure"

2) the graph of $f_X(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$, $t \in \mathbb{R}$



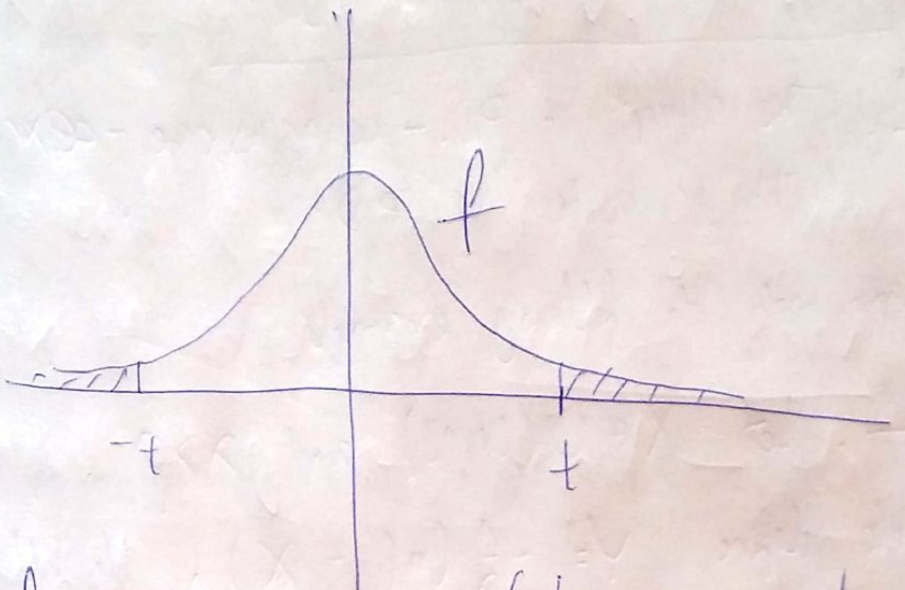
is called „the bell curve"



Then $F_X(t) = P(\{\omega \in \Omega : X(\omega) < t\}) = \int_{-\infty}^{t} f_X(u)\,du$

In particular, $\underline{\Phi(0) = 0.5}$.

(2) Fix $t > 0$ as above



Then for $-t$ we have (by symmetry)

$$\Phi(-t) = P(\{\omega \in \Omega : X(\omega) < -t\})$$

$$\int_{-\infty}^{-t} f_X(u)\,du = \int_{t}^{+\infty} f_X(u)\,du = P(\{\omega \in \Omega : X(\omega) \geq t\})$$

and finally

$$\forall_{t > 0} \quad \Phi(-t) = 1 - \Phi(t)$$

so Φ is determined by positive arguments.

TASK 1

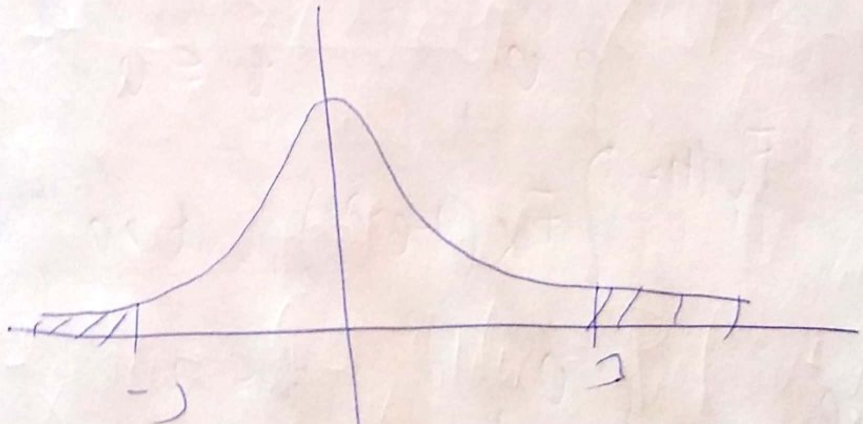
Compute

$$P(\{\omega\in\Omega : |X(\omega)| < t\}), \quad t>0$$

$$X\in N(0,1)$$

4) Later we will show, that for $t_0=3$

$$P(\{\omega\in\Omega : X(\omega)\geq 3\}) < 0{,}01$$



so *almost* all information concerning of $X\in N(0,1)$ is included in $[-3,3]$.

5) In every (well written) book you can find the table of $N(0,1)$.

Ex 2

For $X \in N(-2,1)$, find $P(A)$, where

$$A = \{ \omega \in \Omega : |X(\omega)| < 0{,}5 \}$$

We will use the standarization procedure

$$X \longrightarrow \left.\frac{X-m}{\sigma}\right|_{\substack{m=-2 \\ \sigma=1}} \in N(0,1)$$

$$A = \{ \omega \in \Omega : -0{,}5 < X(\omega) < 0{,}5 \}$$

$$= \{ \omega \in \Omega : \frac{-0{,}5-(-2)}{1} < \underbrace{\frac{X(\omega)-m}{\sigma}}_{= N} < \frac{0{,}5-(-2)}{1} \},$$

so

$$P(A) = P(\{ \omega \in \Omega : 1{,}5 \leq N(\omega) < 2{,}5 \})$$

$$= \Phi(2{,}5) - \Phi(1{,}5)$$

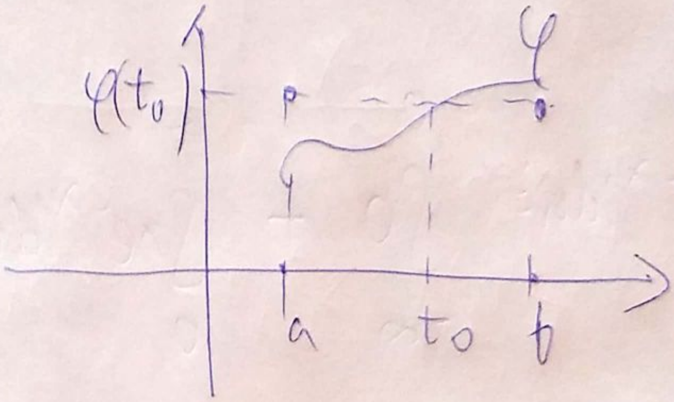
from the TABLE: $0{,}99379$ $\quad$ $0{,}93319$

An introduction to the concept of the parameters

I The concept of the MEAN VALUE.

We begin from the some remark.

Suppose that we have a continuous and positive function φ on finite interval $[a,b]$



From the theory of integral it follows that

$\exists_{t_0 \in (a,b)}$ that $\int_a^b \varphi(t)dt = (b-a)\varphi(t_0)$,

or

$$\varphi(t_0) = \frac{\int_a^b \varphi(t)dt}{b-a}.$$

Then the number $\varphi(t_0)$ is called the MEAN VALUE of φ.

Note, let the measure of the set

$$\{(x,y) \in \mathbb{R}^2 : x \in [a,b],\ 0 \leq y \leq \varphi(x)\}$$

is the same as a rectangle ~~with~~ with the side measure: $b-a$ and $\varphi(x_0)$

Now, we are going to adopt the above idea to the case of probability distribution. So let X be a given R.V, and (Ω, Σ, P) a corresponding K.P.M., so

$$X : \Omega \longrightarrow \mathbb{R}.$$

~~The~~ Analogous to the ~~cl~~ classical ~~MEAN~~ MEAN VALUE PROBLEM we define

Def 2. If there exists the integral $\int_\Omega X dP$, then we say that X has its mean value m or has an expected value EX and we write

$$m = EX = \int_\Omega X \, dP .$$

Th 1 (Linear property).

The operation (as an integral)

$$X \longrightarrow EX$$ has linear property, so:

If X_1, X_2 have mean value, then for all $a, b \in \mathbb{R}$,

$aX_1 + bX_2$ has mean value and we have

$$E(aX_1 + bX_2) = aEX_1 + bEX_2$$

and $Ec = c$ (c – constant dist.).

Now it is a time to explain how to determine
$\int_{\Omega} X dP$ for given R.V. X.

In general it is not so easy, because to give the answer we need to know the general theory of integral — the Lebesgue integral.

For this reasons, we consider only of two types of distributions: discrete and continuous, and only for those cases we will give the answer.