ERASMUS BS course
winter 2020/2021
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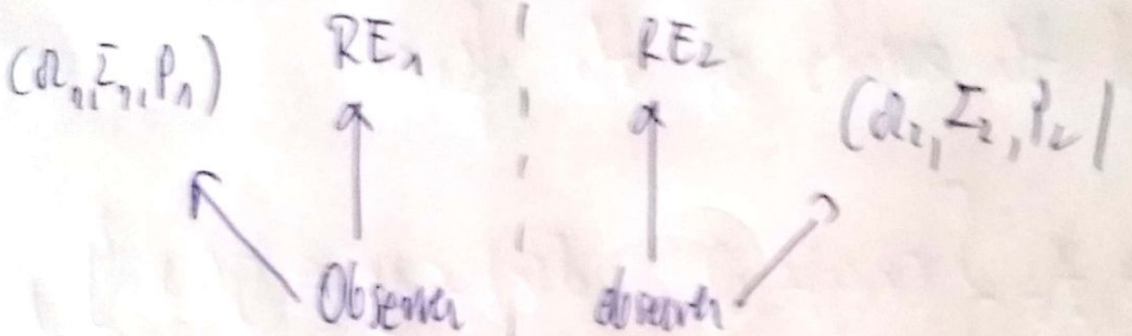
<u>Subject</u>: <u>KPM and its examples — continuation</u>

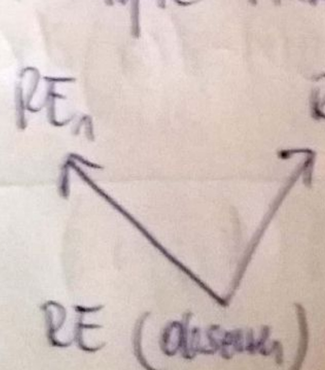
The case of PRODUCT KPM

Suppose that we observe <u>separately</u> two RE: RE_1 & RE_2



So, we have <u>two</u> KPM corresponding to these observations: $(\Omega_1, \Sigma_1, P_1)$ & $(\Omega_2, \Sigma_2, P_2)$.

Let's look at these observations <u>differently</u> — namely, as for an unique random experiment:



Then, we have <u>the new</u> RE with KPM (Ω, Σ, P)

Since RE follows from RE_1, RE_2, the KPM (Ω, Σ, P) should do follow from $(\Omega_1, \Sigma_1, P_1)$, $(\Omega_1, \Sigma_2, P_2)$ as well.

How to discribe (Ω, Σ, P).

The solution comes from the concept of PRODUCT KPM as follow:

$\Omega = \Omega_1 \times \Omega_2$ (cartesian product),

so $\omega \in \Omega \iff \omega = (\omega_1, \omega_2)$, $\omega_j \in \Omega_j$, $j = 1, 2$.

$\Sigma = \Sigma_1 \otimes \Sigma_2$, its means that Σ is the <u>smallest</u> σ-algebra contained the family $\{A_1 \times A_2,\ A_j \in \Sigma_j,\ j = 1, 2\}$.

In particular every subset

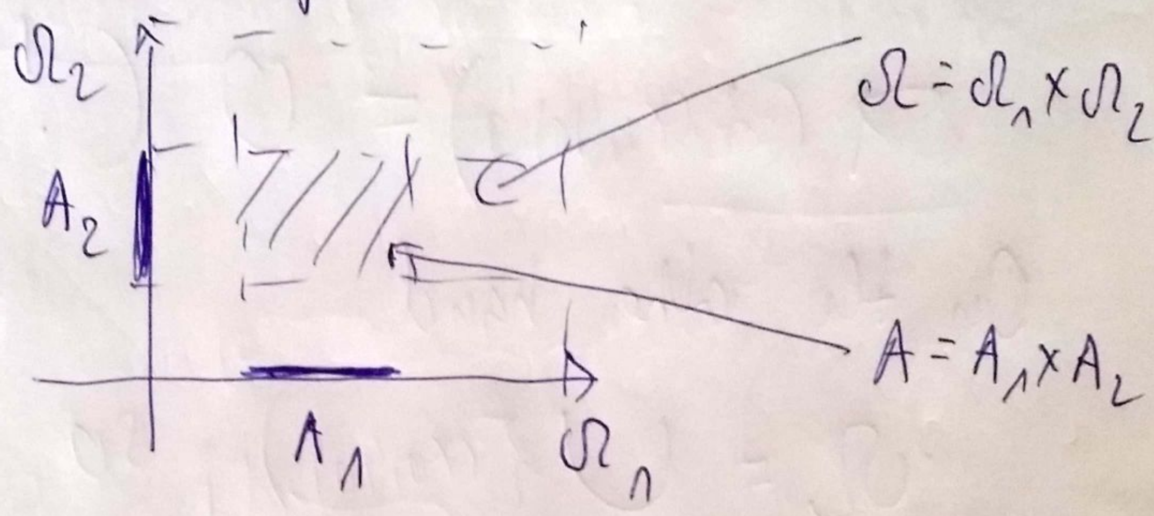
$A_1 \times A_2 \subset \Omega$ $(A_j \in \Sigma_j)$ <u>is an event</u>.

Further, $A_1 \times A_2$ is called a rectangle.

Then

$$P(A_1 \times A_2) \overset{df}{=} P_1(A_1) P_2(A_2).$$

We can give very simple geometrical interpretation of the above facts:



Ex 1. Suppose that

$$(\Omega_1, \Sigma_1, P_1) = (\Omega_2, \Sigma_2, P_2) = (\Omega_0, \Sigma_0, P_0),$$

where $\Omega_0 = \{0, 1\}$, $\Sigma_0 = \mathcal{P}(\Omega_0)$,

$$P_0(\{1\}) = p \in (0,1), \quad P_0(\{0\}) = q = 1 - p.$$

Let's take the cartesian product $(\Omega_0, \Sigma_0, P_0)$ by itself. Then we obtain:

$$\Omega = Bin_2 = \{(0,0), (0,1), (1,0), (1,1)\}$$

$\Sigma = P(\Omega)$,

$P(\{(0,0)\}) = q^2$, $P(\{(0,1)\}) = P(\{(1,0)\}) = pq$,

$P(\{(1,1)\}) = p^2$.

<u>And generally</u>:

for $n \geq 2$, let RE_j $j = 1 \dots n$, and $(\Omega_j, \Sigma_j, P_j)$ <u>are given</u>, so we have

$$\begin{array}{cccc} RE_1 & RE_2 & \cdots & RE_n \\ \uparrow & \uparrow & & \uparrow \\ (\Omega_1,\Sigma_1,P_1) & (\Omega_2,\Sigma_2,P_2) & & (\Omega_n,\Sigma_n,P_n) \\ \text{obs.} & \text{obs.} & \cdots & \text{obs.} \end{array}$$

By using the concept of PRODUCT KPM we get <u>the new</u> (or final result) RE and KPM (Ω, Σ, P), where:

(i) $\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$, so

$\omega \in \Omega \equiv \omega = (\omega_1, \omega_2, \dots, \omega_n)$,

$\omega_j \in \Omega_j$, $j = 1, 2, \dots, n$

(ii) $\Sigma = \Sigma_1 \otimes \Sigma_2 \otimes \dots \otimes \Sigma_n$

is the smallest σ-algebra which contains the family

$$\{ A_1 \times A_2 \times \dots \times A_n , \ A_j \in \Sigma_j , \ j = 1,2 \dots n \}$$

Then $A = A_1 \times A_2 \times \dots \times A_n$ is called a n-rectangle

(iii)

For given A we have

$$P(A) = P_1(A_1) P_2(A_2) \dots P_n(A_n)$$

TASK 1.

Fix $n = 2$. Prove that the family

$$\{ A_1 \times A_2 , \ A_j \in \Sigma_j , \ j = 1,2 \}$$

is not a σ-algebra.

Example 2.

Let's consider Example 1 for the case when $n \geq 2$.

Then we have:

$$\Omega = \Omega_0 \times \Omega_0 \times \ldots \times \Omega_0 = \Omega_0^n = \mathrm{Bin}_n$$

$$\omega \in \Omega \Leftrightarrow \omega = (\omega_1, \omega_2, \ldots, \omega_n),$$

$$\omega_j \in \Omega_0, \; j = 1, 2, \ldots n$$

$$P(\{\omega\}) = p^k (1-p)^{n-k}, \text{ where}$$

$$k = \#\{j : \omega_j = 1\}.$$

Conclusion.

$B(n,p)$ is an example of PRODUCT KPM.

TASK 2.

For $n=2$ let's consider:

$$A = A_1 \times \Omega_2, \quad A_1 \in \Sigma_1$$
$$B = \Omega_1 \times A_2, \quad A_2 \in \Sigma_2$$

Prove that

$$P(A \cap B) = P(A)\,P(B)$$

What does that mean for $A, B \in \Sigma = \Sigma_1 \otimes \Sigma_2$?

Let's go back to the essence of the KPM.

The case of discrete KPM shows that for every an elementary event $\{\omega\}$,

$$P(\{\omega\}) > 0,$$ and always we have

$$\sum_{\omega \in \Omega} P(\{\omega\}) = 1.$$

Is it always like that?

To discuss this problem let's consider the following example of KPM.

Ex).

For: $\Omega = [a_1, b_1] \times [a_2, b_2]$, where $a_1 < b_1$ & $a_2 < b_2$ are reals and the events of the form

$$\forall A = \{ (x,y) \in \Omega : d(x) \leq y \leq g(x) \},$$

where $d, g : [a_1, b_1] \longrightarrow [a_2, b_2]$ are continuous functions, we define

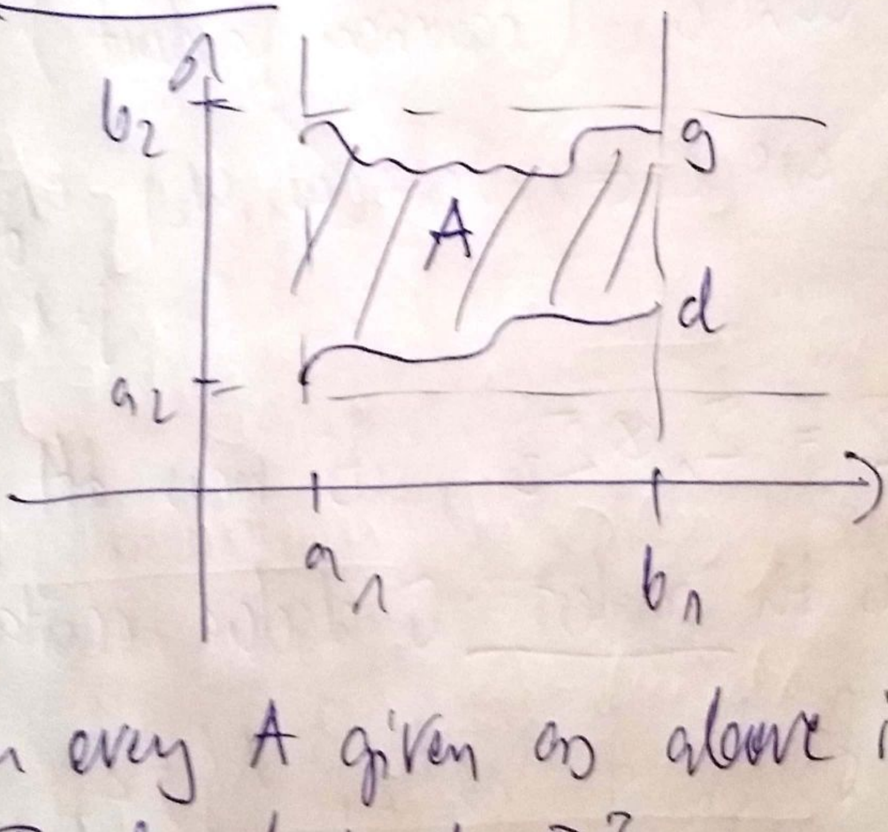
$$P(A) = \frac{\int_{a_1}^{b_1} (g(x) - d(x))\,dx}{(b_1 - a_1)(b_2 - a_2)}.$$

It can be proved that

$(\Omega, \sigma(\mathcal{A}), P)$, where

$\sigma(\mathcal{A})$ - is a σ-algebra generated by the family $\mathcal{A}$ is well defined KPM.

We call this model the geometric model of dimension 2.



then every A given as above is called a Borel subset of $\mathbb{R}^2$

Now, let's take an elementary event, so the point $\{(x_0, y_0)\}$, where $x_0 \in [a_1, b_1]$, $y_0 \in [a_2, b_2]$.

It can be proved that

$$P(\{(x_0, y_0)\}) = 0 \ (!).$$

On the other hand

$$\Omega = \bigcup_{\omega = (x_0, y_0) \in \Omega} \{(x_0, y_0)\}, \text{ so}$$

We have

$$1 = P(\Omega) = P\left(\bigcup_{\omega = (x_0, y_0) \in \Omega} \{(x_0, y_0)\}\right) =$$

$$\overset{*}{=} \sum_{\omega = (x_0, y_0) \in \Omega} P(\{(x_0, y_0)\}) = \sum_{\omega \in \Omega} 0 .$$

Could the math stop working?

Not, everything is alright. So how to explain the above phenomenon (*)?

The answer is as follows:

in (*) the operation Σ is performed in the case then the set Ω is uncountable

For this reason „the sum of zeros" can be equal to 1!

The last conclusion means, that the above KPM is totally undiscrete. Such a model is called CONTINUOUS

TASK 3.

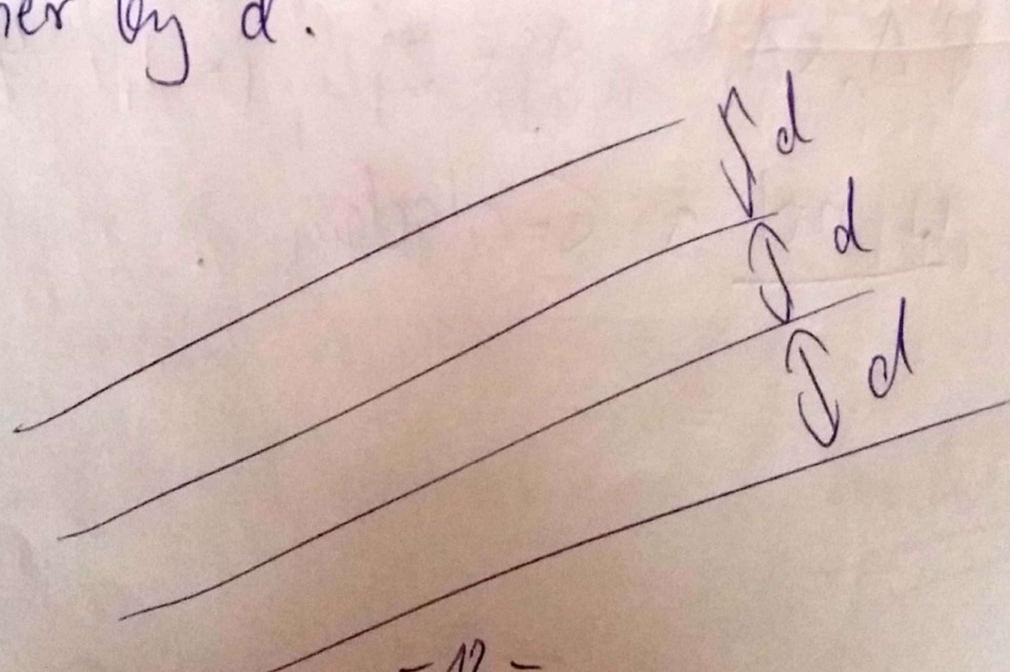
We choose two numbers $p, q \in [-1, 1]$.
Let's calculate the probability that

$$\forall_{x \in \mathbb{R}} \quad x^2 + px + q > 0$$

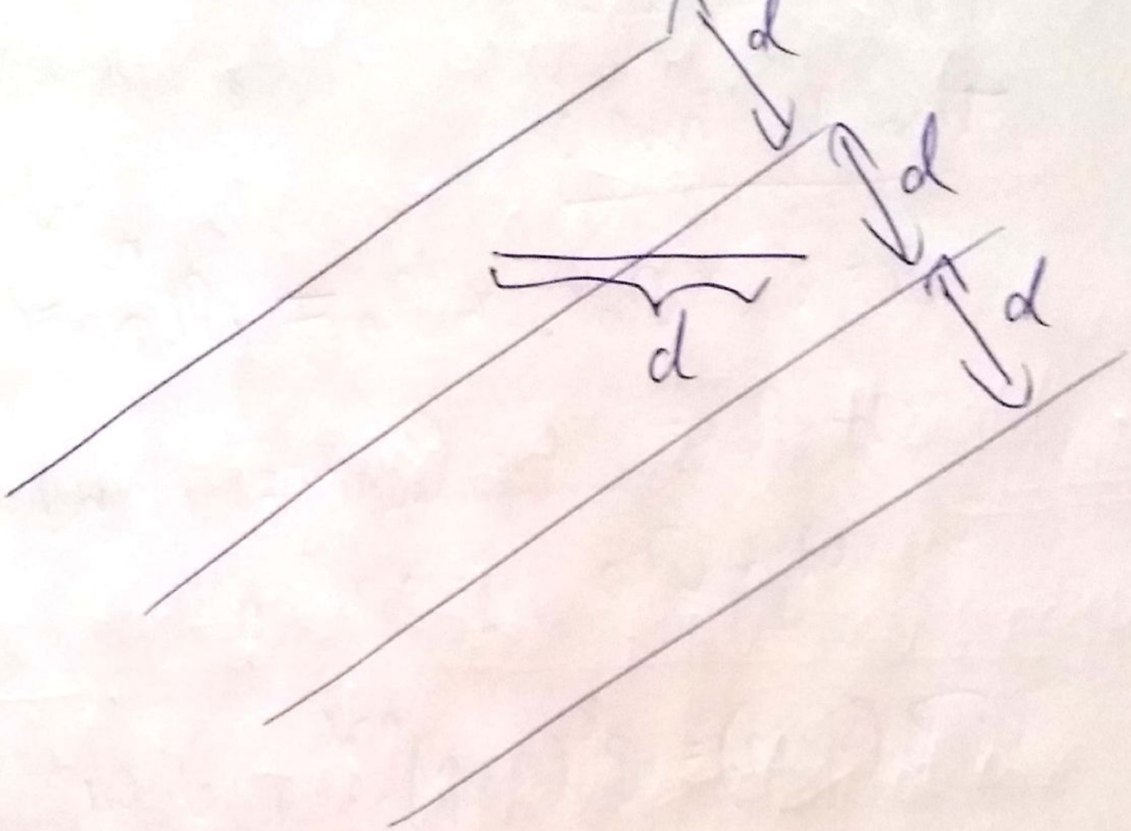
Hint. Use the concept of geometrical model.

TASK 4. (The PROBLEM of BUFFON's needle)

Let's imagine that we have ~~a plane~~ at our disposal the plane on which parallel lines are drawn, distance from each other by d.



We have a needle of length d, which we lower on this plane



Prove that the probability of the event that the lowered needle will cross one of the line is equal to $\frac{2}{\pi}$.

Consider the case then the length of the needle is equal $h \geq d$.

__Hint__. Use the geometrical 2-d model