ERASMUS BA course
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Subject. The concept of probability distribution. (P.D.)

PART I - the case of discrete P.D.

Introduction.

We know that every RE has its description as a KPM

$$RE \longleftrightarrow KPM.$$

In general, we have the following situation:

(i)

$$RE \to KPM_1, \quad RE \to KPM_2, \quad \vdots \quad RE \to KPM_n$$

so given RE we can modeled by using „different" KPM's.

(ii) the sample point $\omega \in \Omega$ does not have to be a real number.

For these reasons the description of RE by KPM should do be expanded.

It can be done using the concept of P.D.

Def 1 (D.P.D. - discrete prob. distribution)

By D.P.D. we understand every function

$$d : A \to [0,1], \text{ where}$$

$A = \{a_n \in \mathbb{R},\ n \in \mathbb{N}_0 \subset \mathbb{N}\}$ - countable

$$\forall_{n \in \mathbb{N}_0} \quad d(a_n) = p_n \ \& \ \sum_{n \in \mathbb{N}_0} p_n = 1.$$

Remarks. 1°. In general, we have infinite sequence (p_n), so $\sum p_n$ means the series and the sum of the series whenever it is converged.

2°. In special case - A is finite, we have

$A = \{a_1, a_2, \ldots, a_n\}$

$d(a_k) = p_k, \ k = 1, 2, \ldots n$, and then we can write d as follows

$$d: \begin{array}{|c|c|c|c|} a_1 & a_2 & \ldots & a_n \\ \hline p_1 & p_2 & \ldots & p_n \end{array}$$

Therefore the table above represents of d.

Examples

1°. Two-pointed distribution

$|A|=2$, $A=\{a_1,a_2\}$

$$\begin{array}{c|c} a_1 & a_2 \\ \hline p & q \end{array} \quad p\in(0,1)$$
$q=1-p$

If $A=\{0,1\}$ and

$$d: \begin{array}{c|c} 0 & 1 \\ \hline q & p \end{array} \quad p\in(0,1) ,$$

then we say that d is standard 2-pointed d.

2°. $B(n,p)$, $n\geq 2$, $p\in(0,1)$

↳ Bernoulli or Binomial distr.

In this case $A=\{0,1,2,\ldots,n\}$

$$A \ni k \longrightarrow p_k = d(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

3°. $P(\lambda)$, $\lambda>0$

↳ Poisson distribution

$A=\{0,1,2\ldots\}$, so is infinite

$$A \ni k \longrightarrow p_k = d(k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

TASK 1

Prove that: $\forall_{k \in A} \; p_k \in [0,1] \;\&\; \sum_{k \geq 0} p_k = 1$

4°.

$$d: \begin{array}{c|c|c|c} -1 & 0 & 2 & 3,5 \\ \hline 0,1 & 0,2 & 0,4 & 0,3 \end{array}$$

Now we show, that given D.P.D d it can be represented by the another way.

To do this we need the concept of cumulative probability function (C.P.F).

Def 2 (C.P.F. in general sens)

Each function F such that

$F: \mathbb{R} \longrightarrow [0,1]$ and satisfies

(i) $\lim_{t \to -\infty} F = 0$, $\lim_{t \to +\infty} F = 1$

(ii) F is non decreasing, so

$\forall_{t_1, t_2 \in \mathbb{R}} \; t_1 < t_2 \Rightarrow F(t_1) \leq F(t_2)$

(iii) F is at least left-side continuous, so

$\forall_{t_0 \in \mathbb{R}} \; \lim_{t \to t_0^-} F = F(t_0)$

is called C.P.F.

Remark

We can meet with different definition of C.P.F., namely in (iii) we can see the right-side continuity condition, so $\lim_{t \to t_0^+} F = F(t_0)$.

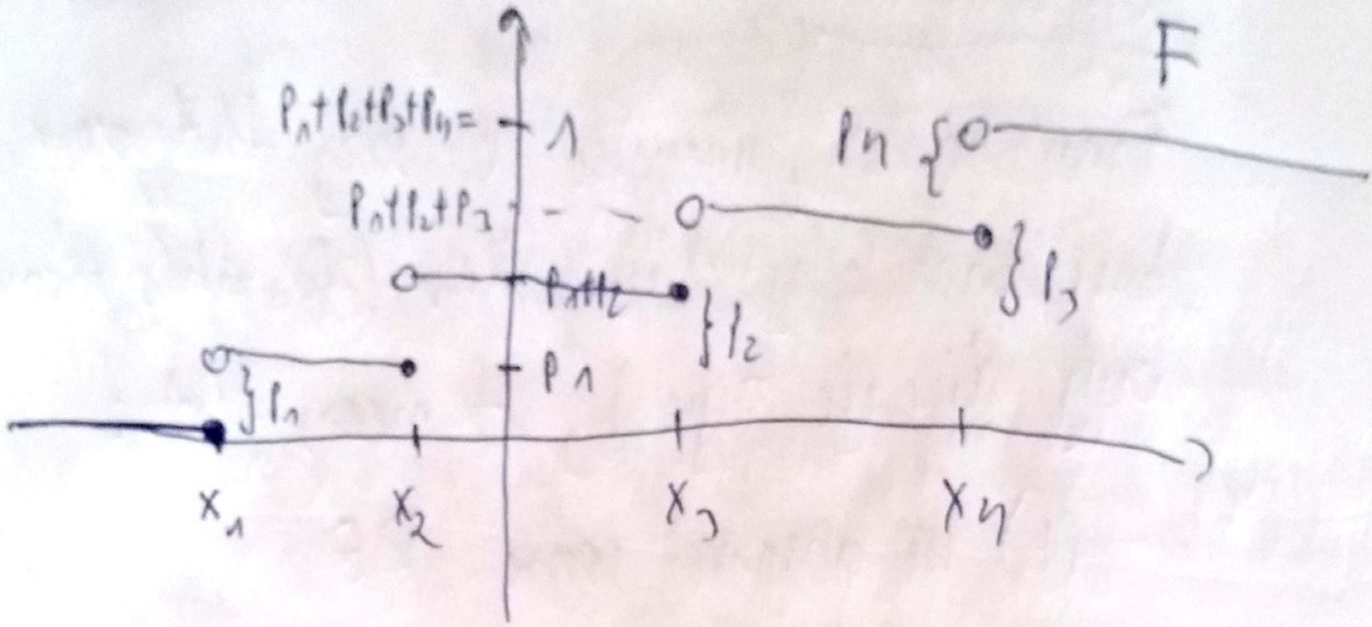
Finally, from perpective of probability theory, those two definitions are equivalent.

Def 3 (D.C.P.F. - discrete C.P.F.)

If in additional, there exist a partition $\{I_j, j \in \mathbb{N}_0 (\mathbb{N})\}$ of $\mathbb{R}$, that on each atom I_j, F is constant, we say that F is D.C.P.F.

Below we show an example of D.C.P.F.

Ex 5



Comment

a) the set $\{x_1, x_2, x_3, x_4\}$ generates the partions $\{I_1, I_2, I_3, I_4, I_5\}$, where

$I_1 = (-\infty, x_1\rangle$, $I_2 = (x_1, x_2\rangle$, $I_3 = (x_2, x_3\rangle$,

$I_4 = (x_3, x_4\rangle$, $I_5 = (x_4, +\infty)$.

b) On each atom I_j, F is constant

c) F has all properties from the def. 2

d) at any point x_j, $j = 1,2,3,4$ we have

$\lim\limits_{t \to x_j^-} F = F(x_j)$, so from (c) - (d)

is at least left-side continuous.

e)

$$\forall_j \quad p_j = \lim_{t \to x_j^+} F - F(x_j) \quad (\text{„jump effect"})$$

Due of the above F is called the „step" function.

So D.C.P.F. is represented by the step-function.

Assumption.

Later we will assume that for given step function F, the number of all jumps is finite.

Let's fix F as above. Then we have:

(i) a set $A = \{x_1, x_2, \ldots, x_n\}$

(ii) $\forall\ k \in \{1, \ldots, n\}$

$$p_k = \lim_{t \to x_k^+} F - F(x_k) \in (0,1] \text{ and}$$

$$\sum_{k=1}^{n} p_k = 1 .$$

It means that we have a function, say d,

$d: A \to [0,1]$, where $d(x_k) = p_k$

This gives us

Th 1. For every step function F given as above there exists a unique discrete prob. distribution d such that for every jump point x_k

$$d(x_k) = p_k = \lim_{t \to x_k^+} F - F(x_k).$$

And vice versa

Th 2. For every D.P.D. d, say with

$$d(a_k) = p_k \quad (k = 1, 2 \dots n)$$

there exists a unique D.C.P.F. F, that (only) at a_k F has its jump with the gape measure equal to

$$p_k = \lim_{t \to a_k^+} F - F(a_k).$$

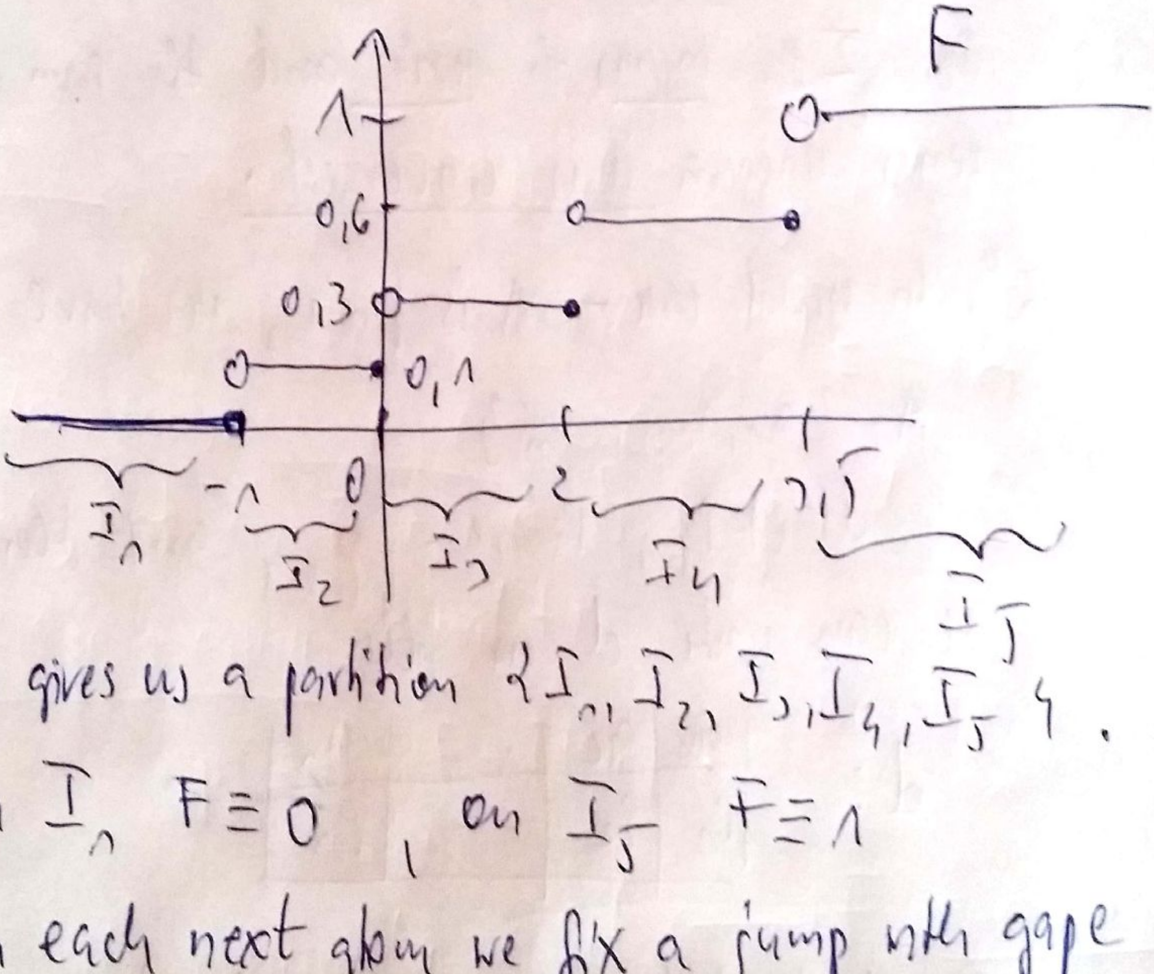
So the correspondence $d \longleftrightarrow F$ is a unique.

Now, by using the example we show the proof of theorem 2.

Let d:

-1	0	2	$3{,}5$
$0{,}1$	$0{,}2$	$0{,}4$	$0{,}3$

Step 1. We arrange in an increasing way the set $\{-1, 0, 2, 3{,}5\}$ on the real line



This gives us a partition $\{I_1, I_2, I_3, I_4, I_5\}$.

On I_1 $F \equiv 0$, on I_5 $F \equiv 1$

On each next atom we fix a jump with gape measure: $0{,}1$, $0{,}2$, ..., $0{,}3$ respectively.

Finally, we obtain the step function F, as above

Let's return to the main subject – RE and its KPM.

We note that formaly in def. of d & F we do not see neither RE nor KPM.

To change it, we need another definitions.

Def 4 (Random Variable in general sens)

We say that we have a random variable (RV) X if:

a) there exist at least one RE and its KPM (Ω, Σ, P)

b) there exist a transformation X with

$$X : \Omega \longrightarrow \mathbb{R},$$

where $\forall_{t \in \mathbb{R}} \{\omega \in \Omega : X(\omega) < t\} \in \Sigma$.

Def 5 (DRV – discrete random variable)

If in addition the set of all values of X is countable, so $X(\Omega) = \{x_n, n \in N_0 \subset N\}$, then X is called DRV.

Let's assume that X is DRV, so

$$X(\Omega) = \{x_n,\ n \in \mathbb{N}_0 \subset \mathbb{N}\}.$$

For every $n \in \mathbb{N}_0$ we put

$$A_n = \{\omega \in \Omega :\ X(\omega) = x_n\}.$$

<u>TASK 2</u>

Prove that the family $\{A_n,\ n \in \mathbb{N}_0 \subset \mathbb{N}\}$ is a partition of Ω.

We define $p_n \overset{df}{=} P(A_n),\ n \in \mathbb{N}_0$.

<u>TASK 3</u>.

Prove that $\sum_{n \in \mathbb{N}_0} p_n = 1$.

So for every $n \in \mathbb{N}_0$ we have

$$a_n \xrightarrow{d} d(a_n) = p_n \in (0,1)$$

with $\sum_{n \in \mathbb{N}_0} p_n = 1$.

Theorem 3

For every DRV X there exists a unique DPD and DCPF: d_X, F_X such that

(*) $\forall_{n \in \mathbb{N}_0} \; P(\{v \in \Omega : X(v) = x_n\}) = d_X(x_n) = \lim_{t \to x_n^+} F - F(x_n)$

Moreover,

(**) $\mathbb{R} \ni t \longrightarrow P(\{v \in \Omega : X(v) < t\}) = F_X(t)$.

It's a bit harder to prove the reverse to Th 3

Theorem 4.

For every d (so and F) there exists at least one DRV, say X that we have (*) and (**) from Th. 3

SUMMARY:

From now on, whenever we say that "we have probability distribution" (for this moment only discrete type), it means that:

① we observe some RE

② we have KPM corresponding the activity

③ we have a transformation

$\Omega \ni \omega \longrightarrow X(\omega) \in \mathbb{R}$, which interpretes each sample point ω as a real number $X(\omega)$ that the (*) and (**) in th. 3 holds.

This part we end by considering the following example.

Ex. 5. X has a distribution as follows

-2	1	3	6
$0,3$	$0,1$	$0,2$	$0,5$

Find the distribution of $Y = -2X + 1$.

By assumption $\exists (\Omega, \mathcal{F}, P)$ such that

$P(A_{-2}) = P(\{\omega \in \Omega : X(\omega) = -2\}) = 0,3$

$P(A_1) = P(\{\omega \in \Omega : X(\omega) = 1\}) = 0,1$

$P(A_3) = P(\{\omega \in \Omega : X(\omega) = 3\}) = 0,2$

$P(A_6) = P(\{\omega \in \Omega : X(\omega) = 6\}) = 0,5$, and

$\forall \omega \in \Omega : Y(\omega) = -2X(\omega) + 1$. Therefore

$B_5 = \{\omega \in \Omega : Y(\omega) = 5\} = A_{-2}$

$B_{-1} = \ldots\ldots = A_1$

$B_{-5} = \ldots \quad A_3$

$B_{-11} = \ldots \quad A_6$.

Finally d_Y:

5	-1	-5	-11
$0,3$	$0,1$	$0,2$	$0,5$

TASK 4

Let X has distribution as in example 5.

Let's consider the events:

$$A_1 = \{\omega \in \Omega : a \le X(\omega) < b\}$$

$$A_2 = \{\omega \in \Omega : a < X(\omega) \le b\}$$

$$A_3 = \{\omega \in \Omega : a < X(\omega) < b\}$$

By using the d_X and F_X let's calculate $P(A_j)$, $j = 1, 2, 3$, where

(i) $a, b \notin \{-2, 1, 3, 6\}$

(ii) $a, b \in \{-2, 1, 3, 6\}$

(iii) $a \in \{-2, 1, 3, 6\}$
$b \notin \{-2, 1, 3, 6\}$.

Hint For every DRV X and DCPF F_X

$$P(\{\omega \in \Omega : X(\omega) \le t_0\}) = \lim_{t \to t_0^+} F_X(t), \quad t_0 \in \mathbb{R}$$

Why?