ERASMUS DG course
Winter 2020/2021
FPM&St, November 12

Subject. The concept of P.D.

$\boxed{\text{PART II - the case of continuous P.D.}}$

We have known that not every KPM is discrete because there are so called continuous models

Therefore, ~~one should expect that~~ it should be expected that apart from discrete distributions there are other distributions

For technical reasons, we will only consider one such type

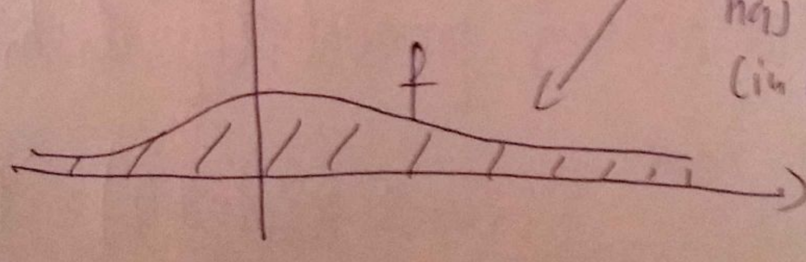
Def 1 (of continuous distribution)

By continuous distribution we understand a function

$f: \mathbb{R} \longrightarrow \langle 0, +\infty)$, which is continuous maybe outside of the finite set

and $\int_{-\infty}^{+\infty} f(t)dt = 1$

Ex 1

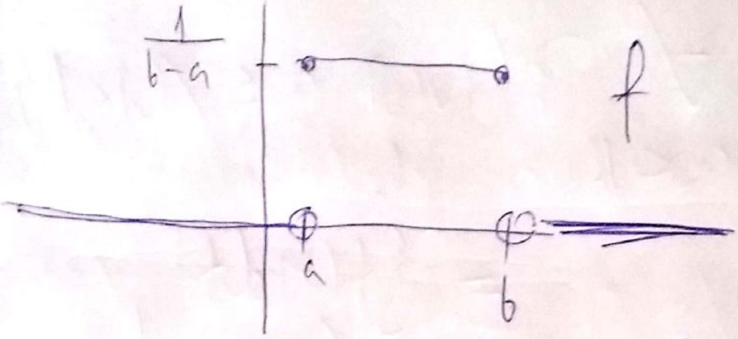


The selected area has a measure (in the sense of integral) equal to one

Ex2

Let's consider the function

$$f(t)=\begin{cases} \frac{1}{b-a}, & t\in[a,b] \\ 0, & t\notin[a,b] \end{cases}$$



We note that only on $\{a,b\}$, so finite set f is discontinuous. Clearly, $f\geq 0$.

By additivity of integral we have

$$\int_{-\infty}^{+\infty} f(t)\,dt = \int_{-\infty}^{a} 0\,dt + \int_{a}^{b} \frac{1}{b-a}\,dt + \int_{b}^{+\infty} 0\,dt = \frac{b-a}{b-a} = 1$$

It means that f represents a continuous distribution.

Later, such distribution is denoted by $U([a,b])$ and is called <u>uniform</u> distribution on interval $[a,b]$

Now we fix a continuous distribution f, and we consider the following function

$$\mathbb{R} \ni t \longrightarrow \int_{-\infty}^{t} f(u)\,du .$$

It is clear that

1°. The value of this function belongs to $[0,1]$ (because $f \geq 0$ and $\int_{-\infty}^{+\infty} f = 1$)

2°. this function is nondecreasing : in fact for $t_1 < t_2$

$$\int_{-\infty}^{t_2} f(u)\,du = \int_{-\infty}^{t_1} f(u)\,du + \int_{t_1}^{t_2} f(u)\,du \;, \text{ and}$$

since $\int_{t_1}^{t_2} f(u)\,du \geq 0$ we get the desired property

3°. this function is <u>differentiable</u> (so smooth) and $\frac{d}{dt}\int_{-\infty}^{t} f(u)\,du = f(t),\quad t \in \mathbb{R}$

what follows <u>from general theory of integral</u>

We have

Th 1 For every continuous distribution f there exists a unique C.P.D.F. F such that

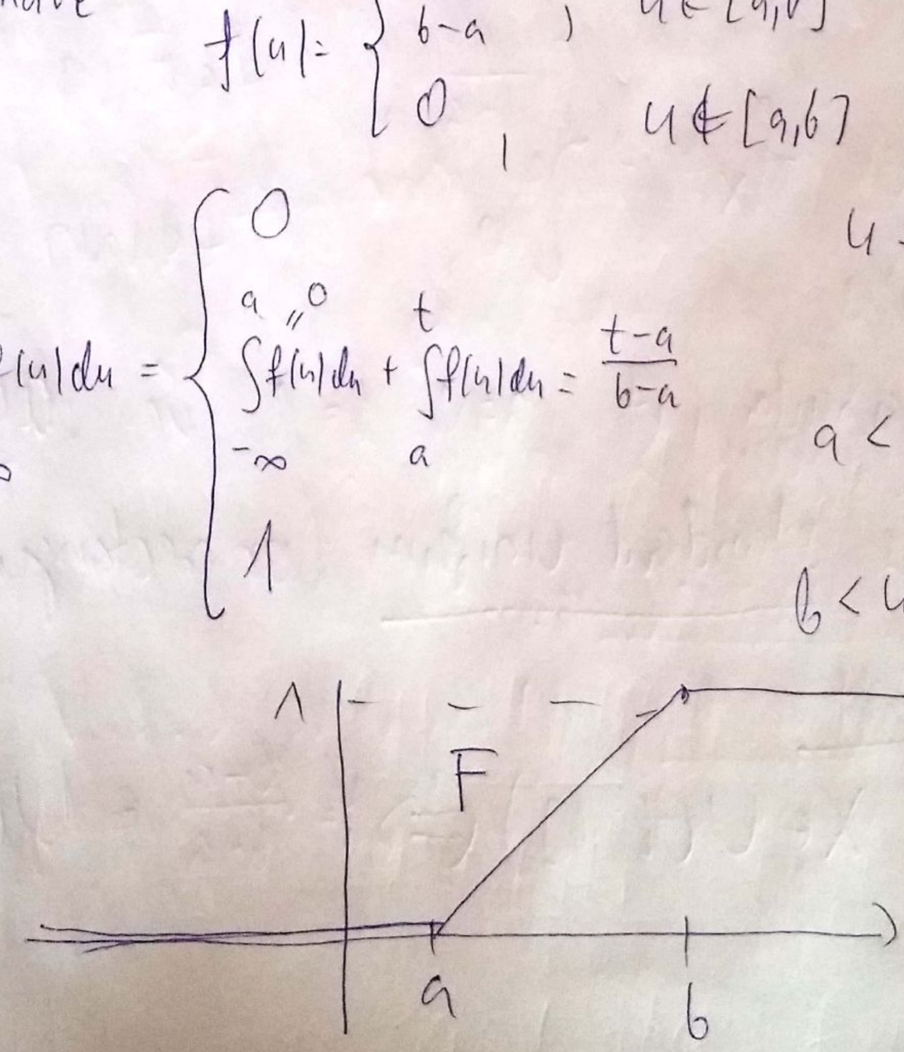
$$\forall_{t \in \mathbb{R}} \; F(t) = \int_{-\infty}^{t} f(u)\,du \text{ and } F'(t) = f(t)$$

ex 3. Find F for f from ex. 2.

We have

$$f(u) = \begin{cases} \frac{1}{b-a}, & u \in [a,b] \\ 0, & u \notin [a,b] \end{cases}$$

$$\int_{-\infty}^{t} f(u)\,du = \begin{cases} 0 & u \le a \\ \underbrace{\int_{-\infty}^{a} f(u)\,du}_{0} + \int_{a}^{t} f(u)\,du = \frac{t-a}{b-a} & a < u \le b \\ 1 & b < u \end{cases}$$



Th2. Suppose that F is C.P.D.F which is a smooth function, so has its F'.

Let's take the function f

$$\mathbb{R} \ni t \xrightarrow{f} F'(t) = f(t).$$

Then f is a continuous distribution.

Indeed, since F is nondecreasing, $f \geq 0$.

Now $F'(t) = f(t),\ t \in \mathbb{R} \iff$

$$\forall a \quad \int_{-\infty}^{a} F'(t)\,dt = \int_{-\infty}^{a} f(t)\,dt$$

But from the main result of the integral calculus

$$\int_{-\infty}^{a} F'(t)dt = \lim_{T \to -\infty} \int_{T}^{a} F'(t)\,dt =$$

$$= \lim_{T \to -\infty} \left(F(a) - F(T)\right) = F(a) - \lim_{T \to -\infty} F(T) = F(a)$$

So we have

$$F(a) = \int_{-\infty}^{a} f(t)\, dt$$

Finally,

$$1 = \lim_{a \to +\infty} F(a) = \lim_{a \to +\infty} \int_{-\infty}^{a} f(t)\, dt = \int_{-\infty}^{+\infty} f(t) dt.$$

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Remark

1° At this moment we have a similar situation as in discrete case – whe have not RE and KPM!
So we need ~~some~~ a tools ~~...~~ to connect the concepts of continuous distribution represented by f and F with RE and KPM.

2°. From now on ~~...~~ we will call f a density function of corresponding continuous probability distribution, F its CPDF.

Def 2 (of continuous random variable)

We say that we have a continuous random variable (C.R.V.) if

1) there exists a RE with KPM (Ω, Σ, P)

2) there exists a function X,

$$X: \Omega \to \mathbb{R} : \underset{t \in \mathbb{R}}{\forall} \{\omega \in \Omega : X(\omega) < t\} \in \Sigma$$

with $X(\Omega) = I$ - for some interval

Th 3

For every C.R.V. X the function

$$\mathbb{R} \ni t \longrightarrow P(\{\omega \in \Omega : X(\omega) < t\})$$

defines a C.P.D.F. corresponding to a continuous probability distribution, so we have $P(\{\omega \in \Omega : X(\omega) < t\}) = F_X(t)$, $t \in \mathbb{R}$

Then $F_X'(t) = f_X(t)$ - the density function corresponding to X.

Moreover, we have

a) $\forall_{a<b}\ P(\{\omega\in\Omega: a \le X(\omega) < b\}) = F_X(b) - F_X(a)$

$$= \int_a^b f_X(t)\,dt,$$

so we can always to calculate the probability by using two methods (F_X & f_X)

b) $\forall_{a\in\mathbb{R}}\ P(\{\omega\in\Omega: X(\omega) \le a\}) =$

$= \lim_{t\to a^+} F_X(t) = F_X(a) = P(\{\omega\in\Omega: X(\omega) < a\})$

and finally

$\forall_{a\in\mathbb{R}}\ P(\{\omega\in\Omega: X(\omega) = a\}) = 0$

and for this reason

$\forall a < b$

$P(\{u \in \Omega : a \leq X(u) < b\}) = P(\{u \in \Omega : a < X(u) \leq b\})$

$= P(\{u \in \Omega : a < X(u) < b\})$

And vice versa

Th. 4 For every continuous probability distribution f (so and F) the exists at least one RE and KPM that the conditions (a) – (b) of the Theorem 3 hold.

Summary

If we say that we have C.R.V. X it means that we include the results of th 3 & 4

Now we can compare the case discrete with continuous

type of distr.	probability distr	C.P.D.F
discrete	d_X	F_X - as a step function
continuous	f_X	F_X - as a smooth f.

$X:$ (the two rows above)

<u>Ex 3</u>. $X \in U([a,b])$, means that

$$f_X(t) = \begin{cases} \frac{1}{b-a}, & t \in [a,b] \\ 0, & t \notin [a,b] \end{cases}.$$

In particular, if $a=0$, $b=1$ we say that X is <u>standard uniform</u> (or rectangular).

<u>Fact</u>

$$X \in U([a,b]) \Leftrightarrow Y = \frac{X-a}{b-a} \in U([0,1])$$

Proof. Let $X \in U([a,b])$. So we have KPM (Ω, Σ, P) and

$$F_X(t) = P(\{\omega \in \Omega : X(\omega) < t\}).$$

By def. $Y(\omega) = \frac{X(\omega) - a}{b-a}$, $\omega \in \Omega$, and

$$F_Y(t) = P(\{\omega \in \Omega : Y(\omega) < t\}) =$$

$$= P\left(\left\{\omega \in \Omega : \frac{X(\omega)-a}{b-a} < t\right\}\right) = P(\{\omega \in \Omega : X(\omega) < a + t(b-a)\})$$

$$= F_X(a + t(b-a)).$$

But

$$f_Y(t) = \frac{d}{dt} F_Y(t) = f_X(a + t(b-a)) \cdot (b-a)$$

(WHY?)

Which prove tt $Y \in U([a,b])$ //

Exh. For $X \in U([0,1])$ find the distribution of

$Y = \ln(X+1)$.

Let (Ω, Σ, P) be a corresponding KPM.

Since $X(\omega) \in [0,1]$ (WHY?)

$X(\omega)+1 \geq 1$, $\omega \in \Omega$, and $Y(\omega) \geq 0$, $\omega \in \Omega$.

It means (WHY?) $F_Y(t) = 0$, $t \leq 0$.

So we can assume that $t > 0$. Then

$F_Y(t) = P(\{\omega \in \Omega : \ln(X(\omega)+1) < t\}) =$

$= P(\{\omega \in \Omega : X(\omega) < e^t - 1\}) = F_X(e^t - 1)$,

so $$F_Y(t) = \begin{cases} 0, & t \leq 0 \\ F_X(e^t - 1), & t > 0. \end{cases}$$

Consequently,

$$f_Y(t) = F_Y'(t) = \begin{cases} 0, & t \leq 0 \\ \frac{d}{dt} F_X(e^t - 1), & t > 0. \end{cases}$$

Show that $\frac{d}{dt} F_X(e^t - 1) = f_X(e^t - 1)e^t$ and

finally $$f_Y(t) = \begin{cases} e^t, & t \in [0, \ln 2] \\ 0, & t \notin [0, \ln 2] \end{cases}$$