ERASMUS B.4 course
winter 2020/2021

Fundamentals of Probabilistics Methods

October 24 by R.R.

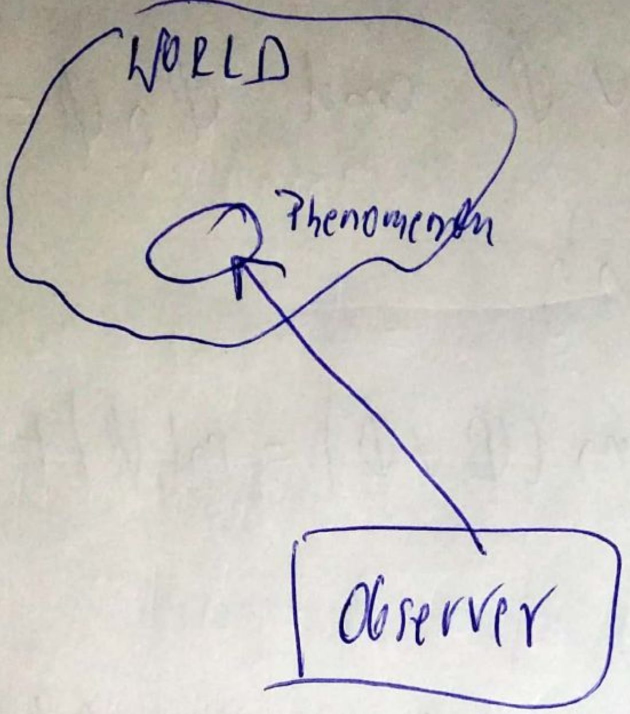
Organizational matters

- self introduction
- form of classes: online / classic ≡ HYBRID SYSTEM
- private website / mail ✓
- passing the course
- the request: In general I am asking for punctuality and participation in classes both in the classical and remote form.

Subject of the lecture

Random phenomenon and its stochastic model

We will start with the definition of a random phenomenon (RP).
What's going on when we think about R.P.
First of all, we will act as an observer of a fragment of reality



For some reasons, we will not be able to uniquely state what is the result of our observation, but we will know what we can notice.

If we adopt this principle of analysis of the phenomenon, we will say it is a RP!

Ex 1. Flipping one coin

When doing this, you agree with me, it is rather difficult to predict its outcome.
On the other hand, we know what to expect — heads (H) or tails (T)
As we will show later, it is much easier

to treat this experiment as a RP than to try to calculate its result.

Ex2. Tossing one die

Ex3. Flipping two, three, etc coins
Tossing two, three, etc. dies

Ex4. A more complex example:
- weather forecast
- quality control
- predicting the course – forecasting the share price
- predicting the winner of the election (Trump or Biden?)
- predicting the effects of a pandemic (COVID-19)

TASK 1.
Please provide other examples of R.P.

We will now focus on the description of R.P that will lead us to the concept of stochastic model (SM)

A concept of SM

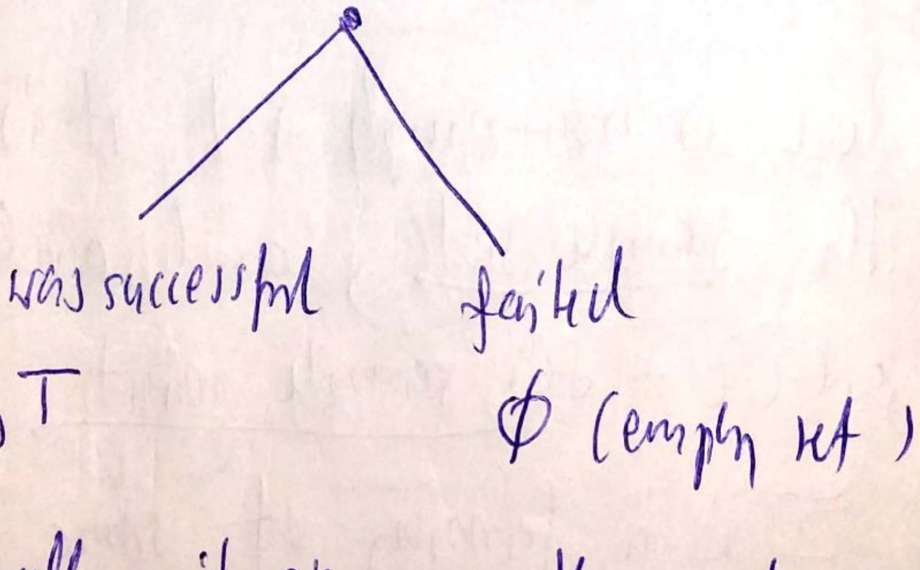
To do this we will use an example #1.
To give a complete description first of all we need to establish the spectrum of all outcomes.
It is clear that we have : H & T.
In mathematics it means that we have a set, say with the name

$$\Theta = \{ H, T \}.$$

$\hookrightarrow$ outcomes

Is that all? No. Why?

If we claim that we have $\mathcal{O}$,
in fact we assume that the experiment
was successful.
However, as we know, it does not have
to be that way.
Therefore, we need to include it in our reasoning.
To do it we need another quality
description, namely such which
interprets well the following situation



Finally, it gives us the next set, say S

$S = \{ W, \{H\}, \{T\}, \emptyset \}$.

We note, that S is a special set – it is a collection of the set.

Further, such set will be called the family

~~Now we have~~

We already have a complete qualitive description of our phenomenon.

To practically use this description we need another level – quantitative description, so some kind of measure.

To be precisely, the measure of elements of the family S, so

$$S \ni s \longrightarrow m(s) \in [a, b]$$

$a < b$, reals.

Of course, such a measure should to have some properties, for example:

(i) $\forall_{S_1, S_2 \in S}$ $S_1 \subseteq S_2 \Rightarrow m(S_1) \le m(S_2)$ (monotonicity)

(ii) $\forall_{S_1, S_2 \in S}$ $S_1 \cap S_2 = \phi \Rightarrow$
$$m(S_1 \cup S_2) = m(S_1) + m(S_2)$$
(additivity property)

You will agree with me that these requirements are self-evident — are clear!

Note, that then it is easy to determine the value of the number a.
Indeed, by monotonicity
$$a = m(\phi), \quad b = m(U).$$

But $\phi = \phi \cup \phi$ and $\phi \cap \phi = \phi$,

so by additivity

$$a = m(\phi) = m(\phi \cup \phi) = m(\phi) + m(\phi) =$$
$$= a + a,$$

and $a = 2a$, which implies that

$$\underline{\underline{a = 0}}.$$

What about the value of b?

What do we do when we see an object for the first time?

We repeat this activity many times!

So let us do it:

we will get a series of the results of our observation:

$(O_1, O_2, O_3, \dots, O_n)$

$\downarrow$ first, $\downarrow$ second, $\downarrow$ the last

How to define the result of such a method?

Simply, by using the notion of the frequence, as a kind of measure, where

$$f_m = \frac{\#\text{number of events noticed}}{\#\text{number of all observ.}}$$

Our event has a name O_j - because $b = m(O)$

In this case we will not either O or ϕ.

That the measure $m(\phi) = 0$, so with no loss of information, if we meet ϕ in O_j we can repeat O_j as long as we ~~obtain~~ only obtain O. Consequently it means, that

In our case we have

$$f_n = \frac{n}{n} = 1,$$

therefore $m(\Omega) = 1$.

In this way we have established the spectrum of m,

$$\underline{\{m(s),\ s \in S\} \subset [0,1]}.$$

Finally, if we put:

$$m(\{H\}) = p \in (0,1)$$

$$m(\{T\}) = 1 - p = q,$$

we get the full (stochastical) description of the RP: <u>flipping one coin</u>, as a object consists of

$$(\Omega, S, m).$$

TASK 2. Repeat the above procedure for example # 2.

TASK 3. What about of $\{x\}$?.

Our next step is a generalization of the above method.

The concept of Kolmogoroff model

Def 1. (of Kolmogoroff model)

By (K.M) we understand the object (Ω, Σ, P), where

(i) Ω is non-empty set, it is called the sample set, and each $\omega \in \Omega$ - the sample point

(ii) Σ is a family of some subsets of Ω, so

$$A \in \Sigma \Rightarrow A \subset \Omega,$$

but in general not vice-versa.

Then A is called - an <u>event</u>,

when Σ has at least the following properties:

a) $\emptyset \in \Sigma$ ("failed" is an event)

b) $A \in \Sigma \Rightarrow A^c \in \Sigma$ (symmetry each)

c) suppose that

$\mathcal{A} = \{A_n, n \in \mathbb{N}_0 \subset \mathbb{N}\} \subset \Sigma$

$\mathbb{N}$ - means positive integers.

$\hookrightarrow$ such family is called <u>countable</u>

Then $\bigcup \mathcal{A} = \bigcup_{n \in \mathbb{N}_0} A_n \in \Sigma$

So the last means that Σ is closed under taken of countable union of events

In particular, for $A, B \in \Sigma$,

$A \cup B \in \Sigma$.

Then we say that Σ is a σ-algebra

(iii) P is a set function with values in $[0,1]$, so

$$P : \Sigma \longrightarrow [0,1],$$

where

$$\Sigma \ni A \longrightarrow P(A) \in [0,1]$$

and P is called the probability function, $P(A)$ - the probability of an event A.

So we have two types of probability:
in more general sense — P, and in a narrow sense — the number $P(A)$.
Please remember this fact!

if P satisfies at least the following conditions:

a) $\forall_{A \in \Sigma} \; P(A^c) = 1 - P(A)$

b) $\forall \mathcal{A} = \{A_n, n \in \mathbb{N}_0 \subset \mathbb{N}\}$

$$\forall_{n \neq m} \; A_n \cap A_m = \emptyset$$

$$P\Big(\bigcup_{n \in \mathbb{N}_0} A_n\Big) = \sum_{n \in \mathbb{N}_0} P(A_n)$$

TASK 4

Suppose s.t. Σ is a σ-algebra.

Prove that:

a) $\Omega \in \Sigma$

b) $\forall_{A, B \in \Sigma} \quad A \cap B \in \Sigma$

c) $\forall_{A, B \in \Sigma} \quad A \setminus B \in \Sigma$

TASK 5

Suppose s.t. (Ω, Σ, P) is KM.

Prove s.t.:

a) $\forall_{A, B \in \Sigma} \quad A \subset B \Rightarrow P(A) \leq P(B)$

b) $\forall_{A, B \in \Sigma} \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

I think that's enough for TODAY!

Thank you for your attention!