ERASMUS B4 course
Winter 2020/2021
FPM&St, November 26

PART I - the parameters of P.D.

Introduction.

We know, that for given R.V. X we can try to perform the operation

$$X \longrightarrow EX = \int_\Omega X dP,$$

which is called the mathematical expectation of X (or equivalently the mean of X or the first moment).
We know the fundamental property of $X \to EX$, namely the linearity property.
It's time to explain how to calculate the integral $\int X dP$.
The problem (of course) is connected to the general theory of integrals — the Lebesgue Integral.
For this reason, I had to reduced our course only to the case when X is discrete or continuous.
As usually we begin with the discrete case.

1°. Discrete case :

Assumption : $X(\Omega) = \{ x_n,\ n \in \mathbb{N}_0 \subset \mathbb{N} \} \subset \mathbb{R}$

$$\forall_{n \in \mathbb{N}_0} \ P(\{\omega \in \Omega : X(\omega) = x_n\}) = p_n \in (0,1)$$

$$\left(\text{so } \sum_{n \in \mathbb{N}_0} p_n = 1 \right).$$

Th 1. $\int_\Omega X dP$ exists iff the series $\sum_{n \in \mathbb{N}_0} p_n x_n$ is converged and we have

$$\int_\Omega X dP = \sum_{n \in \mathbb{N}_0} p_n x_n.$$

△

Remark.

1°. Not every series $\sum_{n \in \mathbb{N}_0} p_n x_n$ is converged! So there exists at least one R.V X for which EX does not exist.

2°. If $|N_0| = k$, so X has a finite P.D.,

then EX always exists, and

$$EX = \sum_{n=1}^{k} p_n x_n .$$

Then the number $\sum_{n=1}^{k} p_n x_n$ is called the

weighted average.

In particular, if X is homogeneous, so

$$p_1 = p_2 = \dots = p_k = \frac{1}{k}, \text{ then}$$

$$EX = \sum_{n=1}^{k} p_n x_n = \frac{1}{k}(x_1 + x_2 + \dots + x_k).$$

This means that EX is an arithmetic mean

Examples:

1°. $d_X : \begin{array}{c|c} x_1 & x_2 \\ \hline q & p \end{array}$, $p \in (0,1)$, $q = 1-p$

$$EX = qx_1 + px_2$$

If X is standard, so $d_X : \begin{array}{c|c} 0 & 1 \\ \hline q & p \end{array}$, $EX = p$.

2°. $X \in B(n,p)$, $n \geq 2$, $p \in (0,1)$.

Since $X(\Omega) = \{0,1,2,\dots,n\}$,

$P(\{\omega\in\Omega : X(\omega) = k\}) = \binom{n}{k} p^k (1-p)^{n-k}$, we have

$$EX = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}.$$

Let's try to calculate EX; we note that

$$EX = \sum_{k=1}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} =$$

$$= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}.$$

Now we do <u>the substitution</u> $k-1=j$.

Then if $k=1,\dots,n \Rightarrow j=0,\dots,n-1$, and

$$EX = \sum_{j=0}^{n-1} \frac{n!}{j!((n-1)-j)!} p^{1+j} (1-p)^{(n-1)-j} =$$

$$= np \underbrace{\sum_{j=0}^{n-1} \frac{(n-1)!}{j!\,((n-1)-j)!}\, p^j (1-p)^{(n-1)-j}}$$

But $\underbrace{(p + (1-p))^{n-1}}_{\parallel} = 1$, so

$$\boxed{EX = np}$$

Remark

We will now show an alternative method of determination EX for $X \in B(n,p)$.

We know that $B(n,p)$ corresponds to R.E which we can describe by using Bernoulli model

$$(\Omega, \Sigma, P) = \underbrace{(\Omega_0, \Sigma_0, P_0) \times (\Omega_0, \Sigma_0, P_0) \times \dots \times (\Omega_0, \Sigma_0, P_0)}_{n\text{-times}}$$

where: 1) $\Omega_0 = \{0, 1\}$, $P_0(\{1\}) = p$ and consequently we have R.V.

$P(\{\omega: X_0(\omega) = 1\}) = p$, $X_0(\Omega) = \Omega_0$

2'. For $X \in B(n,p)$ we have a factorization

$$X = X_1 + X_2 + \dots + X_n,$$

$d(X_j) = d(X_0)$ &

$\{X_1, X_2, \dots, X_n\}$ are statistically independed, so

for every $k \in \{1, 2, \dots, n\}$,

$$P(\{\omega \in \Omega : (X_{n_1}, X_{n_2}, \dots, X_{n_k})(\omega) = (i_1, i_2, \dots, i_{n_k})\})$$

$$= P(\{\omega \in \Omega : X_{n_1}(\omega) = i_1\}) \cdot \ldots \cdot P(\{\omega \in \Omega : X_{n_k}(\omega) = i_{n_k}\})$$

where $i_1, i_2, \dots, i_{n_k} \in \{0, 1\}$

Therefore

$$EX = E(X_1 + X_2 + \dots + X_n) = EX_1 + EX_2 + \dots + EX_n$$

$$= \underbrace{EX_0 + EX_0 + \dots + EX_0}_{n} = n EX_0.$$

But $EX_0 = p$, so $\underline{\underline{EX = np}}$.

3°. $X \in \mathcal{P}(\lambda)$, $\lambda > 0$

Now we have infinite case, so we have to be careful.

First we consider the series: $\sum_{n \in \mathbb{N}_0} p_n x_n$

In our case:

$$\mathbb{N}_0 = \{0, 1, 2 \ldots\}, \text{ so}$$

$$x_n = n \text{ and}$$

$$p_n = P(\{\omega \in \Omega : X(\omega) = x_n\}) = P(\{\omega \in \Omega : X(\omega) = n\})$$

$$= e^{-\lambda} \frac{\lambda^n}{n!}, \quad n \geq 0.$$

Therefore

$$\sum_{n \in \mathbb{N}_0} p_n x_n = \sum_{n \geq 0} n e^{-\lambda} \frac{\lambda^n}{n!}$$ and by definition of series the symbol $\sum_{n \geq 0} n e^{-\lambda} \frac{\lambda^n}{n!}$ means that there exists a limit of the sequence (S_k)

$$S_k = \sum_{n=0}^{k} n e^{-\lambda} \frac{\lambda^n}{n!} \quad , \quad k \geq 1 .$$

We note that

$$S_k = \underbrace{\sum_{n=1}^{k} n e^{-\lambda} \frac{\lambda^n}{n!}}_{\text{an ordinary / arithmetic sum!}} = e^{-\lambda} \sum_{n=1}^{k} \frac{\lambda^n}{(n-1)!}$$

and

$$S_k = \lambda e^{-\lambda} \sum_{n=1}^{k} \frac{\lambda^{n-1}}{(n-1)!} \overset{n-1=j}{=} \lambda e^{-\lambda} \sum_{j=0}^{k-1} \frac{\lambda^j}{j!}$$

But we know (?) that

$$\boxed{\sum_{j=0}^{k-1} \frac{\lambda^j}{j!} \xrightarrow[k \to +\infty]{} e^{\lambda}} \quad , \text{ so}$$

$$S_k \longrightarrow \lambda e^{-\lambda} e^{\lambda} = \lambda .$$

This means that EX exists, and $EX = \lambda$.

TASK 1.

Let X has a finite discrete distribution such that $d_X = d_{-X}$.

Show that $EX = 0$.

TASK 2

Let $m = EX$. Show that $E(X-m) = 0$

TASK 3.

For d_X:

-1	0	1
0,1	0,5	0,4

calculate $E(X^2)$, $(EX)^2$.

TASK 4

We know that $EX = m$. Calculate EY, where $Y = -2X + 5$

TASK 5.

Give an example of discrete p.d. X for which EX does not exist.

Hint: Use the fact that the harmonic series $\sum_{n\geq 1} \frac{1}{n}$ is divergented.

2°. Continuous case.

Assumption: X is a continuous R.V, so

for

$\mathbb{R} \ni t \longrightarrow F_X(t) = P(\{\omega \in \Omega : X(\omega) < t\})$,

$f_X(t) = F_X'(t)$, and

$F_X(t) = \int_{-\infty}^{t} f_X(u)\,du$.

Th2. EX exists ($\int_\Omega X\,dP$) iff there exist ~~and~~ ~~of~~ an (improper) integral

$$\int_{\mathbb{R}} t f_X(t)\,dt = \int_{-\infty}^{+\infty} t f_X(t)\,dt.$$

Remark 1° It means that for every $a \in \mathbb{R}$ there exist $\int_{-\infty}^{a} t f_X(t)\,dt$ and $\int_{a}^{+\infty} t f_X(t)\,dt$,

and

$$\int_{\mathbb{R}} t f_X(t)\,dt = \int_{-\infty}^{a} t f_X(t)\,dt + \int_{a}^{+\infty} t f_X(t)\,dt,$$

where

$$\int_{-\infty}^{a} t f_X(t)\,dt = \lim_{T \to -\infty} \int_{T}^{a} t f_X(t)\,dt$$

$$\int_{a}^{+\infty} t f_X(t)\,dt = \lim_{T \to +\infty} \int_{a}^{T} t f_X(t)\,dt$$

2°. For some X, EX does not exist !

Examples.

1°. $X \in \mathcal{U}([a,b])$.

As we know in this case we have

$$f_X(u) = \begin{cases} \frac{1}{b-a}, & u \in [a,b] \\ 0, & u \notin [a,b]. \end{cases}$$

Let's the examine the integral $\int_{\mathbb{R}} u f_X(u)\,du$

· By additivity property (look at remark 1°) we can write

$$\int_{\mathbb{R}} u f_X(u)\,du = \underbrace{\int_{-\infty}^{a} u f_X(u)\,du}_{I_1} + \underbrace{\int_{a}^{b} u f_X(u)\,du}_{I_2} + \underbrace{\int_{b}^{+\infty} u f_X(u)\,du}_{I_3},$$

whenever the integrals I_1, I_2, I_3 exist. But $I_1 = I_3 = 0$ (WHY?) and

$I_2 = \frac{1}{b-a} \cdot \frac{1}{2}(b-a)^2 = \frac{a+b}{2}$ (WHY). Finally,

$$EX = \frac{a+b}{2} \text{ (in the middle of } [a,b] \text{ !!)}$$