ERASMUS B4 course
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PART II: the parameters of P.D. – the continuous case.

2^0 (as continuation)

Let $X \in N(0,1)$, so

$$f_X(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}, \quad t \in \mathbb{R}.$$

Let's examine the integral $I = \int_{\mathbb{R}} t f_X(t) dt$.

Step 1.

Since $\forall_{t \in \mathbb{R}} \; f_X(-t) = f_X(t)$, if

there exists an integral $\int_0^{+\infty} t f_X(t) dt$,

that there exists an integral $\int_{-\infty}^{0} t f_X(t) dt$

and $\int_{-\infty}^{0} t f_X(t) dt = -\int_0^{+\infty} t f_X(t) dt$.

Consequently, $I = 0$.

Therefore, it is enough to show the existence of $\int_0^{+\infty} t f_X(t) dt$.

Step 2
Let's take

for $T > 0$, $I_T = \int_0^T t f_X(t) dt = \frac{1}{\sqrt{2\pi}} \underbrace{\int_0^T t e^{-t^2/2} dt}_{\tilde{I}_T}$

We will estimate $\tilde{I}_T$ by using "by part" integration method. To this purpose we fix:

$$u = e^{-t^2/2} \qquad dv = t\,dt$$

$$du = -t e^{-t^2/2} dt \qquad v = 1/2\, t^2$$

, then

$$\tilde{I}_T = 1/2\, t^2 e^{-t^2/2} \Big|_0^T + 1/2 \int_0^T t^3 e^{-t^2/2} dt =$$

$$= 1/2\, T^2 e^{-T^2/2} + 1/2 \int_0^T t^3 e^{-t^2/2} dt$$

Now, by d'Hospital rule $\frac{1}{2} T^2 e^{-T^2/2} \xrightarrow[T \to +\infty]{} 0$
(WHY?)

For the integral $\int_0^T t^3 e^{-t^2/2} dt$, from the fact that

$\forall_{u>0}\ e^u > \frac{u^5}{5!}$ we have the following estimation

$$0 \le \int_0^T t^3 e^{-t^2/2} dt \le \int_0^T ?$$

please calculate!

and $\lim_{T \to +\infty} \int_0^T ?$ exists.

Thence $\int_0^{+\infty} t f_X(t) dt$ exists and finally

$$\underline{\underline{EX = 0}}.$$

Theorem.

For every $X \in N(m, \sigma^2)$, EX exists and $EX = m$.

Proof.

Let $X \in N(m, \sigma^2)$. Then by standardization procedure $\frac{X-m}{\sigma} \in N(0,1)$.

By the previous result

$E\left(\frac{X-m}{\sigma}\right) = 0$. But by the linearity property $E\left(\frac{X-m}{\sigma}\right) = \frac{1}{\sigma}EX - \frac{m}{\sigma}$, so

$$\frac{1}{\sigma}EX - \frac{m}{\sigma} = 0 \iff$$

$$EX = m.$$

7*. Let $X \sim E(\lambda)$, $\lambda > 0$, so

$$f_X(t) = \begin{cases} 0, & t \le 0 \\ \lambda e^{-\lambda t}, & t > 0. \end{cases}$$

Check if EX exists?

By definition we have to examine the integral: $I = \int_{\mathbb{R}} t f_X(t)\,dt$.

By the definition of f_X (WHY!)

$$I = \int_0^{+\infty} t \lambda e^{-\lambda t}\,dt = \lim_{T \to +\infty} \lambda \int_0^{T} t e^{-\lambda t}\,dt$$

But for $\tilde{I}_T = \int_0^T t e^{-\lambda t}\,dt = \cancel{-\frac{1}{\lambda} e^{-\lambda t}\Big|_0^T = -\frac{1}{\lambda} e^{-\lambda T} + \frac{1}{\lambda}}$

by „by part rule" we have

$u = t \qquad dv = e^{-\lambda t}\,dt$

$du = dt \qquad v = -\frac{1}{\lambda} e^{-\lambda t}$

$$\frac{\tilde{I}}{T} = -\frac{1}{\lambda} t e^{-\lambda t}\Big|_0^T + \frac{1}{\lambda}\int_0^T e^{-\lambda t}\,dt$$

$$= -\frac{1}{\lambda} e^{-\lambda T} + \frac{1}{\lambda}\int_0^T e^{-\lambda t}\,dt.$$

But $e^{-\lambda T} \xrightarrow[T\to+\infty]{} 0$. and

$$\lambda\int_0^T e^{-\lambda t}\,dt \longrightarrow \int_0^{+\infty} \lambda e^{-\lambda t}\,dt = \int_{\mathbb{R}} f_X(t)\,dt = 1$$

so

$$\frac{\tilde{I}}{T} \xrightarrow[T\to+\infty]{} \frac{1}{\lambda^2}, \text{ and finaly}$$

$$I = \lambda\cdot\frac{1}{\lambda^2} - \frac{1}{\lambda}, \text{ so } \underline{\underline{EX = \frac{1}{\lambda}}}.$$

<u>TASK 1</u>

Find X with continuous distribution that EX <u>does not exists</u>.

<u>Hint</u> Use <u>my materials</u>!

To still understand the role which plays EX in probability theory we need the concept of one another parameter.

First we assume that X is arbitrary RV (it maybe discrete or continuous)

let's take

$$X \longrightarrow Y = X^2$$

$$\downarrow$$

EY if exists.

then the number $EY = E(X^2)$ (is not the same as $(EX)^2$!) is called the second moment of X and we write

$$m_2 = E(X^2) = EX^2$$

Proposition

If EX^2 exists then EX as well, but not vice versa.

Examples

(i) $X(\Omega) = \{x_k,\ k \in N_0 \subset N\}$

$P(\{\omega \in \Omega : X(\omega) = x_k\}) = p_k$

EX^2 exists iff the series $\sum_{k \in N_0} x_k^2 p_k$ is converged.

(ii) $d(X)$ $\begin{array}{c|c} 0 & 1 \\ \hline q & p \end{array}$

$d(X) = d(X^2)$ and therefore $p = EX = EX^2$.

(iii) $X \in B(n,p)$

$$E(X^2) = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k}$$

We have

$$E(X^2) = \sum_{k=1}^{n} k^2 \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} =$$

$$= \sum_{k=1}^{n} k \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}.$$

Now, let $k-1=j \Rightarrow k=1+j$ and

$$= \sum_{j=0}^{n-1} (1+j) \frac{n!}{j!(n-1-j)!} p^{1+j} (1-p)^{n-1-j}$$

$$= \sum_{j=0}^{n-1} \frac{n!}{j!(n-1-j)!} p^{1+j} (1-p)^{n-1-j} + \sum_{j=0}^{n-1} \frac{n!\,j}{j!(n-1-j)!} p^{1+j} (1-p)^{n-1-j}$$

$$\parallel \qquad\qquad\qquad\qquad \parallel$$

$$np \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} p^{j} (1-p)^{n-1-j} \qquad \sum_{j=1}^{n-1} \frac{n!}{(j-1)!(n-1-j)!} p^{1+j} (1-p)^{n-1-j}$$

$$\parallel \qquad\qquad\qquad\qquad \parallel$$

$$np \sum_{j=0}^{n-1} \binom{n-1}{j} p^{j} (1-p)^{n-1-j} \qquad j-1=k$$

$$\parallel \qquad\qquad \sum_{k=0}^{n-2} \frac{n!}{k!(n-2-k)!} p^{k+2} (1-p)^{n-2-k}$$

$$np$$

So finally

$$E(X^2) = np + n(n-2)p^2 \underbrace{\sum_{k=0}^{n-2} \binom{n-2}{k} p^k (1-p)^{n-2-k}}_{1}$$

$$= \underline{\underline{np + n(n-2)p^2}}.$$

TASK 2.

For $X \in P(\lambda)$, calculate $E(X^2)$

Continuous case.

If X is a C.R.V. with

$f_X = F_X'$, then

if there exists $\int_{\mathbb{R}} t^2 f_X(t)\,dt$, then

$$EX^2 = \left(\int_{\Omega} X^2 \,dP \right) = \int_{\mathbb{R}} t^2 f_X(t)\,dt .$$

TASK 7.

Prove that if EX^2 exists, then

$EX^2 \geq 0$.

Examples

(i) $X \in U([0,1])$, so

$$f_X(t) = \begin{cases} 1, & t \in [0,1] \\ 0, & t \notin [0,1] \end{cases}$$

We have:

$$\int_{\mathbb{R}} t^2 f_X(t)\,dt = \underbrace{\int_{-\infty}^{0} t^2 f_X(t)\,dt}_{=0} + \int_0^1 t^2 f_X(t)\,dt + \underbrace{\int_1^{+\infty} t^2 f_X(t)\,dt}_{=0}$$

but: 0 & 0,

so $E(X^2)$ exists and

$$EX^2 = \int_0^1 t^2 f_X(t)\,dt = \tfrac{1}{3}t^3\Big]_0^1 = \underline{\underline{\tfrac{1}{3}}}.$$

TASK 3

Let's take $X \in U([a,b])$.

Show that

$$EX^2 = \frac{(b-a)^2}{12} + \frac{(a+b)^2}{4}$$

by using the scale rule

~~TASK 4~~

Proposition 2

For $X \in N(0,1)$, $E(X^2)$ exists

and $EX^2 = 1$.

TASK 4

Show that for $X \in N(m, \sigma^2)$,

$$EX^2 = \sigma^2 + m^2$$

Hint. Use the standarization property and Prop. 2.