Erasmus BA course
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Subject. Kolmogoroff Probabilistic Model (KPM) and its examples.

Introduction

General Assumptions:

We will further assume that we have a correspondence between given random experiment (RE) and KPM i.e.,

$$RE \longrightarrow KPM = (\Omega, \Sigma, P)$$

and vice versa, i.e.

$$KPM \longrightarrow RE.$$

Unfortunately, the above correspondence usually is not unique. For instance we can meet the following situation:

$$RE \longrightarrow \begin{cases} KPM_1 = (\Omega_1, \Sigma_1, P_1) \\ KPM_2 = (\Omega_2, \Sigma_2, P_2) \\ \vdots \\ KPM_n = (\Omega_n, \Sigma_n, P_n) \end{cases}$$

We will see further how to omit such an uncomfortable relation.

An overview of the selected KPM

1^0. ~~Discrete model case~~
The case of the discrete model (DKPM)
as the first from the historical point of view.

Def We say that KPM (Ω, Σ, P) is discrete if

a) $\Omega = \{\omega_n, n \in \mathbb{N}_0 \subset \mathbb{N}\}$, $\mathbb{N}$ - naturals, $\mathbb{N}_0 \neq \emptyset$

so Ω is countable set

b) $\Sigma = \mathcal{P}(\Omega)$ - the power family, so

$A \in \Sigma \Leftrightarrow$ (iff) $A \in \mathcal{P}(\Omega)$, i.e.

every subset of the sample space Ω is an event

c) $\forall$ ~~every~~ $n \in \mathbb{N}_0$ we know

$P(\{\omega_n\}) = p_n \in (0,1)$, where

$$\sum_{n \in \mathbb{N}_0} p_n = 1.$$

Remarks

1°. If $\mathbb{N}_0$ is finite, then we say that (Ω, Σ, P) is finite.

In this case, if

$$|\Omega| = n, \quad |\Sigma| = 2^{|\Omega|} = 2^n$$

2°. Let $A \in \Sigma$. Then, from c) we have

$$P(A) = \begin{cases} 0, & A = \emptyset \\ \sum_{j \in I_A} p_j, & A = \bigcup_{j \in I_A} \{\omega_j\}, \ I_A \subset \mathbb{N}_0 \end{cases}$$

Indeed, for $A = \bigcup_{j \in I_A} \{\omega_j\}$, $I_A \subset \mathbb{N}_0$, by additivity property of P, we have

$$P(A) = P\left(\bigcup_{j \in I_A} \{\omega_j\}\right) = \sum_{j \in I_A} P(\{\omega_j\}) = \sum_{j \in I_A} p_j$$

3°. If (Ω, Σ, P) is not finite, then we have infinite sequence $(p_n)_{n \in \mathbb{N}_0}$ numbers from $(0,1)$.

Then $\sum_{n \in \mathbb{N}_0} p_n$ denotes the series and its sum, so $\sum_{n \in \mathbb{N}_0} p_n$ is converges to 1!

4°. Suppose that (Ω, Σ, P) is finite and in additional we have

$$p_1 = p_2 = \dots = p_n = p \in (0,1)$$

Then $np = 1$ and therefore $p = \frac{1}{n}$.

In general, for every $A \in \Sigma$ we have

$$P(A) = \frac{|A|}{|\Omega|}$$

It was a first formula on probability function.

It is known as classical probability function

TASK 1

Prove that for finite KPM with

$p_1 = p_2 = \ldots = p_n = p$

(then we say that (Ω, Σ, P) is <u>HOMOGENOUS</u>)

the following formula holds

$$\boxed{P(A) = \frac{|A|}{|\Omega|}}$$

TASK 2.

Apply the above to the case when RE is as follow

(i) tossing a die

(ii) tossing two dies

<u>The two important examples of discrete models</u>

<u>$B(n,p)$ - Bernoulli (or binomial) model</u>,

where $n \geq 2$, $p \in (0,1)$ are given,

for instant $B(100; 0,75)$, where

by definition

$B(n,p) \longrightarrow (\Omega, \Sigma, P)$, where:

(i) $\omega \in \Omega \iff \omega = (\underbrace{b_1, b_2, \ldots, b_n}_{n})$,

$\forall_{1 \le j \le n} \; b_j \in \{0,1\}$,

so ~~each~~ each sample point ω is a binary n-length sequence.

(ii) $P(\{\omega\}) \overset{df}{=} p^k (1-p)^{n-k}$, where

$k = \#\{j : b_j = 1$, where $\omega = (b_1, b_2 \ldots b_n)\}$.

TASK:

Compute the power of Ω for $B(n,p)$.

TASK 4

For given $k \in \{0, 1, 2, \dots, n\}$, let

$$S_k = \{\omega \in \Omega : \#\{j : b_j = 1, \omega = (b_1 \dots b_n)\} = k\}$$

Prove that:

(i) $P(S_k) = \binom{n}{k} p^k (1-p)^{n-k}$

(ii) The family $\{S_k, k = 0, 1, \dots, n\}$ is a partition of Ω, so

$$\forall_{k_1 \neq k_2} \; S_{k_1} \cap S_{k_2} = \emptyset$$

$$\bigcup_{k=0}^{n} P(S_k) = 1$$

Note on application of $B(n,p)$

We consider a RE with KPM (Ω, Σ, P) which is n-repetion of some RE with KPM $(\Omega_0, \Sigma_0, P_0)$, where

$$\Omega_0 = \{\underset{\underset{1}{\shortparallel}}{\text{„TRUE"}}, \underset{\underset{0}{\shortparallel}}{\text{„FALSE"}}\}$$

$$P_0(\{1\}) = p \in (0,1).$$

To obtain the expected result we need one more assumption:

suppose that we have

$$(R_1, R_2, R_3, \ldots, R_j, R_{j+1}, \ldots, R_n)$$

R_j – j^{th} repetition of $(\Omega_0, \Sigma_0, P_0)$

For every $1 \le j \le n-1$, the result of R_j does not affect the results of R_{j+1}, We will later say that the repetition are independent.

Then $(\Omega, \mathcal{Z}, P) \equiv B(n, p)$.

TASK

We throw a symmetrical coin $n = 100$ times independently.

Suppose that our „succes" is „H".

Calculate the probability that as a result of such an experiment a succes will appear at least 3 times.

$\mathcal{P}(\lambda)$ - Poisson model with parameter $\lambda > 0$.

Here we have:

$\Omega = \{0, 1, 2, \ldots\}$

$\Sigma = \mathcal{P}(\Omega)$

For $\omega \in \Omega$, so $\omega = k \in \Omega$

$P(\{k\}) = e^{-\lambda} \frac{\lambda^k}{k!}$, where

e denotes the Euler number.

So $\mathcal{P}(\lambda)$ is not finite discrete model.

TASK 6

Prove that $\sum_{k \geq 0} P(\{k\}) = 1$

TASK 7

Let $A = \{\omega \in \Omega : \omega \geq 3\}$

Let calculate $P(A)$.

Conditional model

As usually we observe some RE

$$\begin{tikzcd}
(RE) \arrow[rr] & & (\Omega, \Sigma, P) \arrow[dl, leftrightarrow] \\
& \text{observer} \arrow[ul] &
\end{tikzcd}$$

Let (Ω, Σ, P) be a corresponding KPM.

Suppose that before we started observing, "someone" gave us an additional information about RE.

It means, that we have:

(•) some event (this additional inf.)

$B \in \Sigma$ with $P(B) > 0$

Now, $\Longrightarrow$ the result of our observation can be presented as follows

$(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$ — the new KPM,
where:

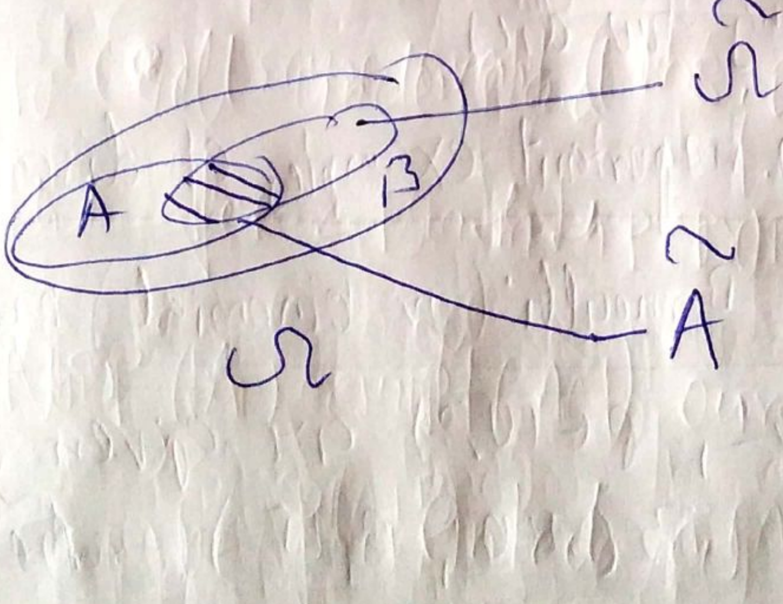
(i) $\tilde{\Omega} = B$

(ii) $\tilde{A} \in \tilde{\Sigma}$ iff $\exists_{A \in \Sigma}$ $\tilde{A} = A \cap B$

$\tilde{A}$ ↓ the "new" event

$A \cap B$ ↓ the "old" event

(iii)
$$\tilde{\Sigma} \ni \tilde{A} \longrightarrow \tilde{P}(\tilde{A}) = \frac{P(A \cap B)}{P(B)}$$



TASK 8.

Prove that (i) $\tilde{\Sigma}$ is an σ-algebra

(ii) the def. of $\tilde{P}$ is correct.

Def: $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$ is called conditional KPM.

Further, $\tilde{P}(\tilde{A})$ is denoted as

$P(A|B)$ and is called

" the probability of A under condition of B "

TASK 9.

When $P(A|B) = P(B|A)$?

TASK 10.

Suppose we have a partition

$\{B_1, B_2, \dots, B_n\}$ of Ω with

$P(B_j) > 0$, $j = 1, 2, \dots, n$.

Prove that for all $A \in \Sigma$

$$P(A) = \sum_{j=1}^{m} P(A|B_j)\, P(B_j).$$

(This is so called formula on <u>total probability</u>)

<u>TASK 11.</u>

Please propose a RE, the analysis of which requires an application of the above formula.

A very important concept is associated with the conditional model, namely the concept of <u>stochastic independence</u>.

To well understand s.i., suppose that for B with $P(B) > 0$, and A

$$P(A) = \tilde{P}(\tilde{A}), \text{ so}$$

$$P(A) = P(A|B) \equiv P(A) = \frac{P(A \cap B)}{P(B)} \equiv$$

$$\boxed{P(A \cap B) = P(A)P(B)}$$

Def

We say that the two events A, B are stochastically independent, if

$$P(A \cap B) = P(A) P(B)$$

TASK 12.

Suppose that A, B are independent.

Prove that a) A^c, B

b) A^c, B^c

are also s.e.

When A is s.i. with A?

We will further show that independence in $B(n,p)$ is exactly equal to s.e.