ERASMUS+, Course: FPM
Winter Semester, 2021/2022

SUPLEMENT to the TOPIC
PARAMETERS OF Probability distributions -
15 December

The notion of the variance

Let X be a R.V. with the second moment, so EX exists as well.

Def (of variance). (s.c.m.)
By the second central moment of X we understand the number

$$E(X-EX)^2$$

second
moment central

Prop 1. If X has a second moment, then second central moment exists. Furthermore

$$E(X-EX)^2 = m_2 - m^2, \quad m = EX$$

Proof.

Since $(X-EX)^2 = X^2 - 2(EX)X + (EX)^2$, by linearity of the operation of E we have

$$E(X-EX)^2 = E(X^2 - 2(EX)X + (EX)^2) =$$

$$= E(X^2) - 2(EX)(EX) + (EX)^2 =$$

$$= E(X^2) - (EX)^2 = m_2 - m^2$$

The s.c.m. of X is also denoted as $var(X)$ and is called the Variance of X.

Ex 1. Let X $\begin{array}{c|c} 0 & 1 \\ \hline q & p \end{array}$

We know that: $m = p$, $m_2 = p$, so

$$var(X) = m_2 - m^2 = p - p^2 = p(1-p) = pq$$

Ex 2. Let $X \in P(\lambda)$, $\lambda > 0$.

We know that $m = \lambda$ and $m_2 = \lambda^2 + \lambda$, hence

$var(X) = \lambda$

Ex 2. Let $X \in B(n,p)$

In this case $EX = np$ and

$$EX^2 = npq + (np)^2,$$ hence

$$\underline{var(X) = npq}.$$

Now we show a different method the calculation of $var(X)$ for $X \in B(n,p)$. To do this, we need the following theorems.

Th 1. Suppose that X has a variance.

then:

① $var(X) \geq 0$

② $var(X) = 0$ iff $X = const$

③ $\forall_{c \in \mathbb{R}}$ $var(cX) = c^2 var(X)$.

TASK 1. Prove the ③:

Th2. Let X & Y have the variances.

If X & Y are stochastically independent, then

$$var(X+Y) = var(X) + var(Y).$$

TASK 2

Prove the th 2.

Hint If X & Y are s. ind., then

$$E(XY) = (EX)(EY).$$

Now we are to show that for $X \in B(n,p)$,

$$var(X) = npq.$$

By factorization theorem for $B(n,p)$,

$X = X_1 + X_2 + \dots + X_n$, where

(i) $d(X_i) = d(X_0)$, $X_0 \; \frac{0}{q} \Big| \frac{1}{p}$,

so $var(X_i) = var(X_0) = pq$

(ii) $X_1, X_2 \dots, X_n$ are stochastically independent

Finally, by th. 2

$$\operatorname{var}(X) = \operatorname{var}(X_1 + X_2 + \dots + X_n) =$$

$$= \operatorname{var}(X_1) + \operatorname{var}(X_2) + \dots + \operatorname{var}(X_n) =$$

$$= \underbrace{\operatorname{var}(X_0) + \operatorname{var}(X_0) + \dots + \operatorname{var}(X_0)}_{n\text{-times}} = npq$$

Ex 4. Let $X \in E(\lambda)$ $(\lambda > 0)$.

Since $EX = \frac{1}{\lambda}$, $EX^2 = \frac{2}{\lambda^2}$,

$$\operatorname{var}(X) = \frac{1}{\lambda^2}$$

Ex 5. Let $X \in U([a,b])$

In this case, $EX = \frac{a+b}{2}$, $E(X^2) = \frac{(b-a)^2}{12} + \frac{(a+b)^2}{4}$

so $\operatorname{var}(X) = \frac{(b-a)^2}{12}$

Ex 6. Let $X \in N(m, \sigma^2)$.

We know that $EX = m$, $EX^2 = \sigma^2 + m^2$, so

$$\operatorname{var}(X) = \sigma^2$$