Kurs: PSK

Forma zajęć - Lab.

Tryb: On-line

L 3

<u>Temat</u> Rozkłady ciągłe.

Problemy, zadania:

(1) X ma rozkład ciągły. Znaleźć rozkład $Y = X^2$

(2) Sprawdzić, że $f(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0, \end{cases}$ $(\lambda > 0)$

jest f. gęstości p. rozkładu.

(3) Pokazać, że $X \in N(m, \sigma^2)$, to $\frac{X-m}{\sigma} \in N(0,1)$

(4) Zad. 5.2.6

(5) Zad. 5.2.7

(6) Zad. 5.2.11

(7) Zad. 7.1.9

(8) Zad. 7.2.6

① Z założenia dla

$$F_X(t) = P(\{\omega\in\Omega : X(\omega) < t\}),$$

$$f_X(t) = F_X'(t), \quad t\in\mathbb{R}.$$

Wyznaczy najpierw F_Y, gdzie $Y = X^2$. Ponieważ

$$\forall_{\omega\in\Omega} \quad Y(\omega) = (X(\omega))^2 \geq 0, \quad F_Y(t) = 0 \text{ dla } t\leq 0.$$

Niech dalej $t>0$. Wtedy

$$F_Y(t) = P(\{\omega\in\Omega : Y(\omega) < t\}) = P(\{\omega\in\Omega : X^2(\omega) < t\})$$

$$= P(\{\omega\in\Omega : |X(\omega)| < \sqrt{t}\}) =$$

$$= P(\{\omega\in\Omega : -\sqrt{t} < X(\omega) < \sqrt{t}\}).$$

Ale X ma rozkł. ciągły, zatem

$$F_Y(t) = P(\{\omega\in\Omega : -\sqrt{t} \leq X(\omega) < \sqrt{t}\}) =$$

$$= F_X(\sqrt{t}) - F_X(-\sqrt{t}).$$

Stąd

$$F_Y(t) = \begin{cases} 0, & t\leq 0 \\ F_X(\sqrt{t}) - F_X(-\sqrt{t}), & t>0. \end{cases}$$

oraz dla f. gęstości mamy:

$$f_Y(t) = \begin{cases} 0, & t \le 0 \\ f_X(\sqrt{t})\frac{1}{2\sqrt{t}} - f_X(-\sqrt{t})\cdot\left(-\frac{1}{2\sqrt{t}}\right) \end{cases}$$

$$= \begin{cases} 0, & t \le 0 \\ \frac{1}{2\sqrt{t}}\left(f_X(\sqrt{t}) + f_X(-\sqrt{t})\right), & t > 0 \end{cases}$$

② Zacznijmy z podanymi tw. wysły sprawdzić, iż

(i) $f(t) \ge 0$, $t \in \mathbb{R}$ – jasne

(ii) $\int_{-\infty}^{+\infty} f(t)dt = 1$.

Ale $\int_{-\infty}^{+\infty} f(t)dt = \underbrace{\int_{-\infty}^{0} f(t)dt}_{=0} + \lambda \int_0^{+\infty} e^{-\lambda t}dt =$

$$= \lambda \lim_{T \to +\infty} \int_0^T e^{-\lambda t}dt = \lambda \lim_{T\to+\infty} -\frac{1}{\lambda}e^{-\lambda t}\Big|_0^T$$

$$= \lambda \lim_{T\to+\infty}\Big[\underbrace{-\frac{1}{\lambda}e^{-\lambda T}}_{\xrightarrow{T\to+\infty} 0}+\frac{1}{\lambda}\Big] = \frac{\lambda}{\lambda} = 1$$

(3) Niech

$$(*) \quad f_X(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-m)^2}{2\sigma^2}}, \quad t\in\mathbb{R}$$

Dla $Y = \frac{X-m}{\sigma}$ mamy

$$F_Y(t) = P(\{\omega\in\Omega : Y(\omega) < t\}) =$$

$$= P\left(\left\{\omega\in\Omega : \frac{X(\omega)-m}{\sigma} < t\right\}\right) =$$

$$= P(\{\omega\in\Omega : X(\omega) < t\sigma + m\}) =$$

$$= F_X(t\sigma + m)$$

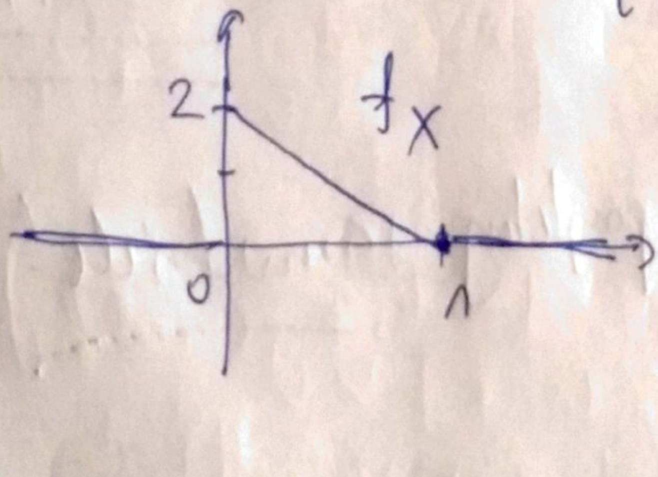
Stąd $f_Y(t) = F'_Y(t) = f_X(t\sigma+m)\cdot\sigma$.

Podstawmy do $\circledast$ $t \to t\sigma + m$

$$f_Y(t) = \frac{\sigma}{\sqrt{2\pi}\,\sigma} e^{-\frac{(t\sigma+m-m)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}} e^{-t^2/2},$$

co dowodzi, że dla Y: $m=0$, $\sigma=1$, w $(\circledast)$, zatem $Y \in N(0,1)$.

(4) Dla X, $f_X(t) = \begin{cases} 2-2t, & t \in [0,1) \\ 0, & t \notin [0,1) \end{cases}$



Stąd

$$P(\{\omega \in \Omega : -1 < X(\omega) < 1/2\}) =$$

$$= P(\{\omega \in \Omega : -1 \leq X(\omega) < 1/2\}) = \int_{-1}^{1/2} f_X(t)\,dt$$

Dokończ!

II sposób:

wyznaczyć F_X i wtedy $F_X(1/2) - F_X(-1)$

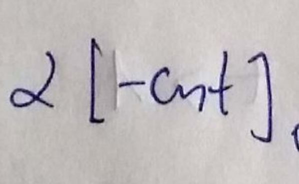
(5)
$$f(t) = \begin{cases} \alpha \sin t, & t \in [0, \pi] \\ 0, & t \notin [0, \pi] \end{cases}$$

Musi być: (i) $f(t) \geq 0 \Rightarrow \alpha > 0$

(ii) Dla takich α, $\int_{-\infty}^{+\infty} f = 1$

Ale $$\int_{-\infty}^{+\infty} f = \underbrace{\int_{-\infty}^{0} 0}_{0} + \int_{0}^{\pi} \alpha \sin t \, dt + \underbrace{\int_{\pi}^{+\infty} 0}_{0}$$

$$= \alpha \left[-\cos t\right]_0^{\pi} = \alpha(-\cos\pi + \cos 0) = 2\alpha$$



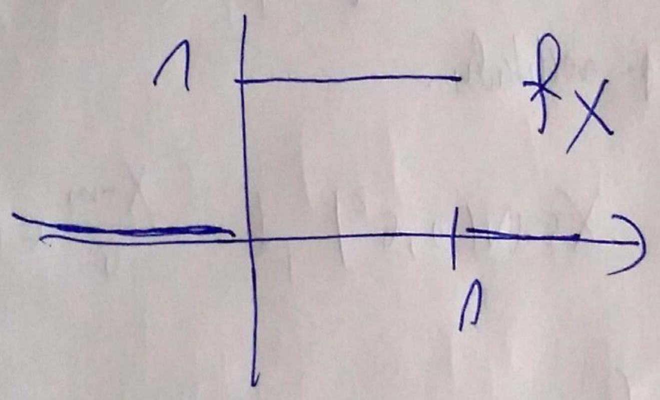
$2\alpha = 1 \Rightarrow \alpha = 1/2$

Dlahu

$$F_X(t) = \int_{-\infty}^{t} f_X(u)\,du$$

Dokonči!

⑥ Nech $X \in \mathcal{U}([0,1])$



Nech $Y = \frac{1}{X+1}$ (porovnajte samostatne!)

$F_Y(t) = P(\{\omega \in \Omega : Y(\omega) < t\}) =$

$= P(\{\omega \in \Omega : \frac{1}{X(\omega)+1} < t\}) =$

$= P(\{\omega\in\Omega: 1 < tX(\omega) + t\})$

$= P(\{\omega\in\Omega: \frac{1-t}{t} < X(\omega)\})$,

gdy $t > 0$, bowiem dla $t \leq 0$

$$P(\{\omega\in\Omega: Y(\omega) < t\}) = 0$$

(bo $Y(\omega) = \frac{1}{X(\omega)+1} \geq 0$, $X(\omega) \in [0,1]$).

Dlatego

$$F_Y(t) = 1 - P(\{\omega\in\Omega: X(\omega) \leq \frac{1-t}{t}\})$$

$$= 1 - P(\{\omega\in\Omega: X(\omega) < \frac{1-t}{t}\})$$

$$= 1 - F_X\left(\frac{1-t}{t}\right)$$

$$F_Y(t) = \begin{cases} 0, & t \leq 0 \\ 1 - F_X\left(\frac{1-t}{t}\right) & t > 0 \end{cases}$$

Stąd

$$f_Y(t) = \begin{cases} 0, & t \leq 0 \\ -f_X\left(\frac{1-t}{t}\right)\left(\frac{1-t}{t}\right)' \end{cases}$$

$$\left(\frac{1-t}{t}\right)' = \frac{-t-(1-t)}{t^2} = -\frac{1}{t^2},$$

skąd

$$f_Y(t) = \begin{cases} 0, & t \leq 0 \\ \frac{1}{t^2} f_X\left(\frac{1-t}{t}\right), & t > 0 \end{cases}$$

Aby określić $f_X\left(\frac{1-t}{t}\right)$, niech $u = \frac{1-t}{t}$

i wyznaczmy dla jakich t, $0 \leq u \leq 1$: $(t>0$

$$0 \leq \frac{1-t}{t} \quad \wedge \quad \frac{1-t}{t} \leq 1$$

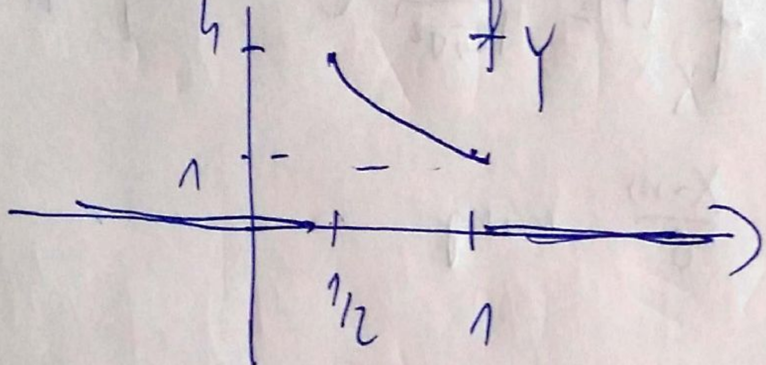
$$\Updownarrow$$

$$t \leq 1 \qquad 1-t \leq t$$

$$1 \leq 2t \qquad t \geq 1/2$$

Ostatnie

$$f_Y(t) = \begin{cases} 0, & t \le 0 \\ \frac{1}{t^2}, & t \in \langle 1/2, 1 \rangle \\ 0, & t \in (0, 1/2) \cup (1, +\infty) \end{cases}$$



(7) $X \in N(-2, 4)$

Oblicz $P(A)$,

$A = \{\omega \in \Omega : -2 < X(\omega) < 7\}$

Mamy: (i) $\frac{X-m}{\sigma} \in N(0,1)$, $m = -2$, $\sigma = 2$

oraz

$$P(A) = P(\{\omega \in \Omega : -2 \le X(\omega) < 7\})$$

$$= P\left(\left\{\omega \in \Omega : \frac{-2-m}{\sigma} \le \frac{X-m}{\sigma} < \frac{7-m}{\sigma}\right\}\right)$$

$$\frac{-2-m}{\sigma} = \frac{-2+2}{2} = 0, \qquad \frac{7-m}{\sigma} = \frac{9}{2} = 4{,}5$$

$$= \Phi(4{,}5) - \Phi(0)$$

Korzystamy z tablic (!) $\mathcal{N}(0,1)$.

Pozostałe przykłady samodzielnie!

⑧ Nech $Y \in \mathcal{W}(\lambda)$, čiže

$$f_Y(t) = \begin{cases} 0, & t \le 0 \\ \lambda e^{-\lambda t}, & t > 0 \end{cases}, \quad \lambda > 0$$

Známy vzťah $X = 1 - e^{-\lambda Y}$!

Zrejme, že $0 \le X(\omega) = 1 - e^{-\lambda Y(\omega)} =$

$$= 1 - \frac{1}{e^{\lambda Y(\omega)}} \le 1,$$

$\lambda Y(\omega) > 0$

Zrejme $X(\omega) \in \langle 0, 1 \rangle$, $\omega \in \Omega$.

Odtiaľ $F_X(t) = 0 \quad t \le 0,$

$F_X(t) = 1, \quad t > 1$

Nech ďalej $t \in (0, 1\rangle$.

$F_X(t) = P(\{\omega \in \Omega : X(\omega) < t\}) =$

$$= P(\{\omega\in\Omega:\ 1 - \frac{1}{e^{\lambda Y(\omega)}} < t\})$$

$$= P(\{\omega\in\Omega:\ 1 - t < \frac{1}{e^{\lambda Y(\omega)}}\})$$

$$= P(\{\omega\in\Omega:\ e^{\lambda Y(\omega)} < \frac{1}{1-t}\})$$

(bo $t\in(0,1)$!)

$$= P(\{\omega\in\Omega:\ \lambda Y(\omega) < \ln(\frac{1}{1-t})\})$$

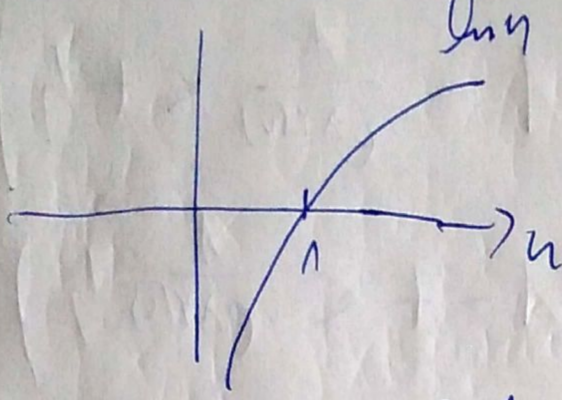
$$= P(\{\omega\in\Omega:\ Y(\omega) < -\frac{1}{\lambda}\ln(1-t)\})$$

$$= F_Y(-\frac{1}{\lambda}\ln(1-t)), \quad t\in(0,1)$$

Skąd

$$f_X(t) = f_Y(-\frac{1}{\lambda}\ln(1-t))\cdot(-\frac{1}{\lambda})\frac{1}{1-t}\cdot(-1)$$

$$= f_Y(-\frac{1}{\lambda}\ln(1-t))\frac{1}{\lambda(1-t)}$$



$u = 1 - t \in (0,1)$

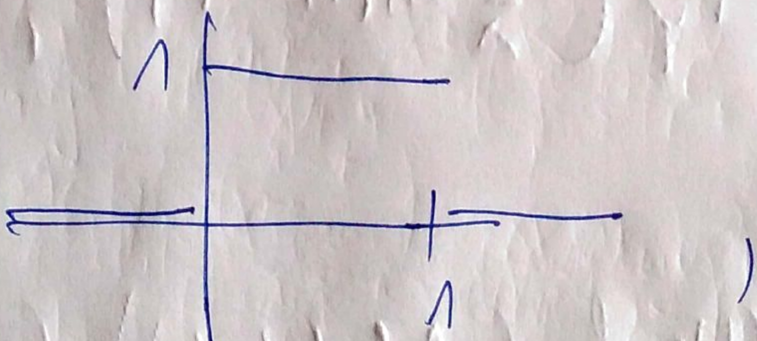
Zatem $-\frac{1}{\lambda}\ln(1-t) > 0$, $t \in (0,1)$.

Dalej

$$f_X(t) = \cancel{\lambda} e^{-\lambda\left(-\frac{1}{\lambda}\ln(1-t)\right)} \frac{1}{\cancel{\lambda}(1-t)}$$

$$= \frac{1}{1-t} e^{\ln(1-t)} = \frac{1-t}{1-t} = 1$$

Zatem



czyli $X \in J((0,1))$