Statistics for engineers

First List of Problems
Introduction to Probability Theory

1. By using the basic property for *probability function*, namely the equation

$$P(E \cup F) = P(E) + P(F) - P(E \cap F), \text{ for } E, F \in \Sigma,$$

find the formula on $P(A \cup B \cup C)$ for arbitrary events $A, B, C \in \Sigma$.

2. Consider an *random experiment* consisting of *independent trials* where each trial can result in one of two possible *outcomes* (for example, *success* or *failure*). We will also assume that the probability of *success* remains constant from trial to trial; we will denote this probability by p where $0 < p < 1$. Such an random experiment is sometimes referred to as *Bernoulli trials.* We can define several random variables from a *sequence of Bernoulli trials.* For example, consider an experiment consisting of n Bernoulli trials. The *sample space* Ω can be represented as all possible 2^n sequences of *successes* (S) and *failures* (F):

$$\Omega = \{F \cdots F, SF \cdots F, FSF \cdots F, \cdots, S \cdots S\}.$$

We can define a random variable $\mathbf{X}$ that counts the number of *successes* in the n Bernoulli trials. Find the *distribution function* of $\mathbf{X}$. Show the relationship of the distribution function and the *Newton binomial formula.*
Hints:

- determine the probability of each outcome in Ω,
- count the number of outcomes with exactly k *successes.*

3. Suppose that $\Omega = \bigcup_{j=1}^{n} B_j$, with $B_k \cap B_l = \emptyset$ for $k \neq l$ and $P(B_j) > 0$, $j = 1, 2, \ldots, n$. Show that for every event A the following equation holds (so called *Law of Total Probability*)

$$P(A) = \textstyle\sum_{j=1}^{n} P(A|B_j)P(B_j).$$

4. Suppose that a random variable $\mathbf{X}$ has a *density function*

$$f(x) = \begin{cases} kx^3, & \text{for } 0 \leqslant x \leqslant 1; \\ 0, & \text{otherwise,} \end{cases}$$

where k is some positive constant. Determine the value of k and then the *cumulative distribution function* of $\mathbf{X}$.

5. Suppose that the random variable $\mathbf{X}$ is a continuous with the density function

$$f_{\mathbf{X}}(x) = \begin{cases} |x|, & \text{for } -1 \leqslant x \leqslant 1; \\ 0, & \text{otherwise.} \end{cases}$$

Define a new random variable $\mathbf{Y} = \mathbf{X}^2$. Find $f_{\mathbf{Y}}$ and then $E\mathbf{Y}$.

6. Show that for $\mathbf{X} \in U([a, b])$, $var(\mathbf{X}) = \frac{(b-a)^2}{12}$.

7. Prove that $\frac{\mathbf{X}-m}{\sigma} \in N(0, 1)$, whenever $\mathbf{X} \in N(m, \sigma^2)$.

8. Suppose that $\mathbf{X}$ has a *standard normal distribution*. Show that $E\mathbf{X}^k = 0$ if k is odd.

9. Suppose taht $\mathbf{X_1}, \mathbf{X_2}, \ldots, \mathbf{X_{100}}$ are independent and identically distributed with $\mathbf{X}_j \in U([0, 1])$. By using *Central Limit Theorem* (CLT) approximate probability of an event

$$\{\omega \in \Omega\colon\ 49 \leqslant \textstyle\sum_{j=1}^{100} \mathbf{X}_j(\omega) < 62\}.$$

10. Let $\mathbf{X} \in B(n, p)$. Show that CLT gives the following approximation rule

$$P(\{\omega \in \Omega\colon\ a \leqslant \mathbf{X}(\omega) < b\}) \approx \mathbf{\Phi}\Big(\tfrac{b-np}{\sqrt{np(1-p)}}\Big) - \mathbf{\Phi}\Big(\tfrac{a-np}{\sqrt{np(1-p)}}\Big),$$

where $p \in (0, 1)$ and $\mathbf{\Phi}$ denotes cumulative distribution function of the *standard normal* random variable.