The concept of the series

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In the theory of real functions the series play a significant role. On the whole, this notion refers to: *series of numbers*, *series of functional*, *trygonometric series.* We assume that we will continue to occupy only the series of number that we call shortly *series*. To do this, further we need the concept of the *sequence of numbers.*

Definition 1 *By a sequence of numbers we understand the real valued function with a domain as a subset of natural numbers.*

For reasons of historical, sequences still will be denote as $(a_n)_{n\in\mathbf{N}_o}$, which means that the function mentioned in the above definition has the following form

$$\mathbf{N}_o \ni n \longrightarrow a_n \in \mathbf{R}, \text{ for } \mathbf{N}_o \subset \mathbf{N}.$$

We assume that we will be further dealt only infinite sequences, so $\mathbf{N}_o = \mathbf{N}$.

Example 1 *If for every $n \geq 1$, $a_n = \frac{1}{n^\alpha}$, for some fixed $\alpha \in \mathbf{R}$, we say that the sequence (a_n) is α–harmonic.*

Example 2 *The sequence (a_n) with $a_n = q^n$ for some $q \in \mathbf{R}$ is called geometric.*

Example 3 *The sequence (a_n) with $a_n = (-1)^n$ is called alternating.*

Definition 2 *Let's given be a sequence $(a_n)_{n\in\mathbf{N}}$. If for every n we take*

$$S_n = a_1 + a_2 + \ldots + a_n = \sum_{i=1}^{n} a_i, \text{ where } S_1 = a_1,$$

we get a new sequence $(S_n)_{n\in\mathbf{N}}$, which is called the series. Then, S_n–the n^{th} expression of those sequence is called the n^{th} partial sum.

Proposition 1 *Every sequence is series and vice versa.*

Proof. According to definition 2 enough to show that each sequence is series. For sequence (b_n) we define the sequence (a_n) as follows

$$a_1 = b_1,\ a_n = b_n - b_{n-1},\ n \geq 2.$$

We note that for every $n \geq 2$

$$b_n = a_1 + a_2 + \ldots + a_n, \text{ and } b_1 = a_1,$$

which means that b_n is the partial sum of the series.

$\square$

Remark 1 *By proposition 1 we see that there is no difference beetwen the notion of sequence and series. Why then do we use the concept of a series? The answer is trivial–first at all from historical reasons.*

Example 4 *If for (S_n), where $S_n = \sum_{i=1}^{n} a_n$ and $a_n = \frac{1}{n^\alpha}$, we say we have α–harmonic series. In particular, for $\alpha = 1$ we get harmonic series.*

The example of *α–harmonic series* shows that in case of series there is problem of determining the value of the partial sum of series. On the other side, from the theory of series follows that it is not necessary for the purpose of understanding the asymptotic behavior of the series known of S_n. For this reason, adopted another way to writing the series (S_n) as

$$(S_n) = \sum_{k=1}^{\infty} a_k, \text{ where } S_n = \sum_{i=1}^{n} a_n.$$

Example 5 *By geometrical series we mean $\sum_{k=1}^{\infty} q^k$. It is worth noting that in this situation, it is not difficult to designate the value of the partial sum. Indeed, we have*

$$S_n = q + q^2 + \ldots + q^n = q\frac{1-q^n}{1-q}, \textit{ if } q \neq 1.$$

For mathematics, the most important thing is to investigate the asymptotic properties of series.

Definition 3 *We say that the series $\sum_{k=1}^{\infty} a_k$ is convergent, if the sequence (S_n) has his limit $S \in \mathbf{R}$. Then S is called the sum, and we write*

$$S = \sum_{k=1}^{\infty} a_k.$$

So, in the case of convergent series the symbol $\sum_{k=1}^{\infty} a_k$ has a double meaning–is a series and the sum of this series.

Remark 2 *For given series $\sum_{k=1}^{\infty} a_k$, we can talk about three cases:*

1. *series is convergent to his sum S,*
2. *series has unproper limit, which we write as*

$$\sum_{k=1}^{\infty} a_k = +\infty \text{ or } -\infty.$$

   Then we say that series is divergent to infinity.

3. *series is not convergent, which means that the sequence (S_n) has not a limit.*

Example 6 *It is well known that the α–harmonic series are convergent for all $\alpha > 1$, but for $0 < \alpha \leq 1$ are divergent.*

Problem 1 *Show that harmonic series is divergent.*

Example 7 *The geometric series is convergent for $|q| < 1$. For $q \geq 1$ is divergent and for $q \leq -1$ is not convergent. Indeed, from example 5 we know that for $q \neq 1$, we have*

$$S_n = q + q^2 + \ldots + q^n = q\frac{1 - q^n}{1 - q}.$$

So, if $|q| < 1$, then q^n tends to 0, and therefore $S_n \longrightarrow S = \frac{q}{1-q}$. If $q \geq 1$, then we can write

$$S_n = q + q^2 + \ldots + q^n \geq n,$$

therefore $S_n \longrightarrow +\infty$. The case $q \leq -1$ is more complicated, and we omit it.

In the theory of series the important role plays so called *criteria for convergence* of series. Mathematics knows many of such criteria. We pay attention only to the basics.

Proposition 2 *If series $\sum_{k=1}^{\infty} a_k$ is convergent then $a_n \longrightarrow 0$.*

Proof. We note that $a_n = S_n - S_{n-1}$, so if $S_n \longrightarrow S$, then also $S_{n-1} \longrightarrow S$, and consequently $a_n \longrightarrow 0$.

□

Remark 3 *The example of the harmonic series show that the condition $a_n \longrightarrow 0$ is not sufficient to make a series convergent.*

Proposition 3 *Suppose that $a_n > 0$ for all natural n. If $\frac{a_{n+1}}{a_n} \longrightarrow p$, and $p < 1$ then series $\sum_{k=1}^{\infty} a_k$ is convergent, if $p > 1$ is divergent. In the case $p = 1$ anything can happen: the series can be divergent but also convergent.*

Example 8 *Let's consider the series* $\sum_{n=0}^{\infty} \frac{1}{n!}$. *By using the above proposition we see that*

$$\frac{a_{n+1}}{a_n} = \frac{1}{n+1} \longrightarrow 0,$$

therefore $\sum_{n=0}^{\infty} \frac{1}{n!}$ *is convergent. It is well known that in this case*

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e,$$

where e *is the Euler number.*

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, than we can write

$$S = S_n + r_n, \text{ where } r_n = \sum_{k=n+1}^{\infty} a_k,$$

and is called the n^{th}–*tail* of this series. Hence, we have the following property

Proposition 4 *If the series is convergent, than the sequence of his tails* (r_n) *is convergent to zero.*

□

If in addition, $a_n \geq 0$ for every n, then the sequence (S_n), as increasing, has always a limit, may be in appropriate terms, so we can write

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} (S_k + a_{k+1} + a_{k+2} + \ldots + a_n) = S_k + \lim_{n\to+\infty} (a_{k+1} + a_{k+2} + \ldots + a_n),$$

and finally

$$\lim_{n\to+\infty} S_n = S_k + r_k, \text{ for every } k.$$

Therefore, from proposition 4 we have another useful criterion

Proposition 5 *For every series with* $a_n \geq 0$ *the following conditions are equivalent*

- $\sum_{n=1}^{\infty} a_n$ *is convergent;*
- *the sequence of tails* (r_k) *is convergent to zero.*

□

Example 9 *By alterning series we mean the series given by alterning sequence, so $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$. We prove that aletrning series is convergent. To do this let's fixe n and consider S_n. Then, depending on whether n is even or odd we get*

$$S_n = b_1 + b_2 + \ldots + b_n, \text{ or } S_n = b_1 + b_2 + \ldots + b_n + \frac{1}{2n+1},$$

where $b_k = \frac{1}{k} - \frac{1}{k+1} > 0$. But $b_k = \frac{1}{k(k+1)} < \frac{1}{k^2}$, and the 2-harmonic series is convergent, therefore by proposition 5, the sequence of tails of series (S_n) is convergent to zero. This means that alternig series is convergent.

Remark 4 *The alterning series is an example of the larger classe of series which are called the Leibnitz series. Moreover, it can prove that in the case of alterning series we have $S = \ln 2$.*

Remark 5 *If the series $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} |a_n|$ is disconvergent, we say that $\sum_{n=1}^{\infty} a_n$ is conditionally convergent. In the theory of conditionally convergent series, the well known theorem is famous Riemann's theorem. The mentioned theorem says that in the case of conditionally convergent series $\sum_{n=1}^{\infty} a_n$, for every real number r we have $r = \sum_{n=1}^{\infty} b_n$, where the sequence (b_n) is given as a permutation of the sequence (a_n). In particular, from example 6 and 9 and the Riemann theorem, we can write $\pi = \sum_{n=1}^{\infty}(-1)^{f(n+1)}\frac{1}{f(n)}$ for a bijection (permutation) $f\colon \mathbf{N} \to \mathbf{N}$ of the set of all naturals.*

Now suppose we have two sequences (a_n) and (b_n) such that $\lim\limits_{n\to+\infty} \frac{a_n}{b_n} = 1$ and we known that the series $\sum_{n=1}^{\infty} a_n$ is convergent. By assumption $|\frac{a_n}{b_n} - 1| < \varepsilon$, whenever $n \geq n_o$ for a natural n_o. But then $1 - \varepsilon < \frac{a_n}{b_n} < 1 + \varepsilon$ for $n \geq n_o$, which means that for those $n's$, $a_n b_n > 0$. Further, without loss of generality we can assume that both, a_n, b_n are positive for $n \geq n_o$. To prove that $\sum_{n=1}^{\infty} b_n$ is convergent it suffices to show that the sequence of partial sum (S_n) is upper bounded. Indeed, for $n \geq n_o$ we have $S_n = S_{n_o} + (b_{n_o+1} + \ldots + b_n)$ and the sum in the parenthesis defined the increasing sequence, which is convergent whenever is upper bounded. But from the previous inequality $0 < \frac{a_n}{1+\varepsilon} < b_n < \frac{a_n}{1-\varepsilon}$ for $n \geq n_o$, which yields

$$0 < b_{n_o+1} + \ldots + b_n < \frac{1}{1-\varepsilon}(a_{n_o+1} + \ldots + a_n) \leq \frac{1}{1-\varepsilon}\Big(\sum_{n=1}^{\infty} a_n - (a_1 + \ldots + a_{n_o})\Big),$$

which prove that (S_n) is upper bounded. From the above we can obtain (why?) the following resuls

Proposition 6 *For given sequences (a_n) and (b_n), let $\lim\limits_{n\to+\infty} \frac{a_n}{b_n} = 1$. Then*

- *$\sum_{n=1}^{\infty} a_n$ is convergent iff $\sum_{n=1}^{\infty} b_n$ is convergent;*

- $\sum_{n=1}^{\infty} a_n$ *is disconvergent iff* $\sum_{n=1}^{\infty} b_n$ *is disconvergent.*

□

Example 10 *We analyse the asymptotic behavior of the series* $\sum_{n=1}^{\infty} \sin \frac{\pi}{n}$. *To do this, we use proposition 6. For this purpose take the harmonic series, which as we known is disconvergent. Since*

$$\lim_{n \to +\infty} \frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}} = 1,$$

and $\pi \sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{\pi}{n} = \infty$, *hence* $\sum_{n=1}^{\infty} \sin \frac{\pi}{n} = \infty$.

In the theory of real functions the important rule play so called *expansion of the functions* given in the form of series. The formulas given below are associated with a number of mathematicians, including: Newton, Leibnitz, Euler, Gauss, Taylor, Maclaurin, Lagrange and others.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \ x \in (-1, 1)$$

$$\sum_{n=0}^{\infty} (-1)^{n+1} x^n = \frac{1}{1+x}, \ x \in (-1, 1)$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x, \ x \in \mathbf{R}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \ln(1+x), \ x \in (-1, 1\rangle$$

$$\sum_{n=1}^{\infty} (-1)^{2n-1} \frac{x^{2n-1}}{(2n-1)!} = \sin x, \quad \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x, \ x \in \mathbf{R}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x, \ x \in \langle -1, 1\rangle.$$

For example, substituting in the formula fourth $x = 1$ we obtain mentioned in remark 4 equation $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = \ln(2)$. Similarly, if in the last formula we assume that $x = 1$, we get the famous *Leibnitz series* $\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$, the first which gave expand the number of π. Of course, from the formula of three we have well known expansion of the Euler's number $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.